New Derived Symmetries of some Hyperkähler Varieties

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Abstract

We construct new autoequivalences of the derived categories of the Hilbert scheme of n points on a K3 surface and the variety of lines on a smooth cubic 4-fold. The second example and $n = 2$ in the first use the theory of spherical functors; to deal with $n > 2$ in the first example we develop a theory of $\mathbb{P}$-functors. We conjecture that the same construction yields an autoequivalence for any moduli space of sheaves on a K3 surface.

In an appendix we give a cohomology and base change criterion which is well-known to experts, but not well-documented.

0 Introduction

This paper grows out of the following observation: Let S be a complex K3 surface, $S^{[2]}$ the Hilbert scheme of pairs of points on S, thought of as a moduli space of ideal sheaves, $F : D^b(S) \to D^b(S^{[2]})$ the functor induced by the universal sheaf $\mathcal{U}$ on $S \times S^{[2]}$, and R the right adjoint of F. Then we will see that the composition $RF = \mathrm{id} \oplus [-2]$, so F is an example of the "spherical functors" of Rouquier [39] and Anno [1] and hence determines an autoequivalence T of $D^b(S^{[2]})$. Briefly, $T = \mathrm{cone}(FR \to \mathrm{id})$. Spherical functors generalize Seidel and Thomas's spherical objects [40] and unify various family versions of them [20, 42]. In §1 we give a simplified definition of spherical functors, review the known examples, and give an alternate proof that they yield autoequivalences in preparation for our work on $\mathbb{P}$-functors in §3.

The autoequivalence T is induced by a shift of

$$\mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^*\mathcal{U}, \pi_{23}^*\mathcal{U}) \in \mathrm{Coh}(S^{[2]} \times S^{[2]}), \tag{0.1}$$

where π_{ij} are the projections from $S^{[2]} \times S \times S^{[2]}$. This sheaf appeared in Markman's work [33] on the Beauville-Bogomolov form. It is a reflexive sheaf of rank 2, locally free away from the diagonal. Thus T sends the structure sheaf of a point to a rank 2 sheaf, so it is not in the subgroup of $\operatorname{Aut}(D^b(S^{[2]}))$ generated by shifts, line bundles, automorphisms of $S^{[2]}$, and $\mathbb{P}$-twists, all of which preserve rank (up to sign). We will also see that it does not come from any known spherical twist on S via the map $\operatorname{Aut}(D^b(S)) \hookrightarrow \operatorname{Aut}(D^b(S^{[2]}))$ studied by Ploog [38].

Next we ask what happens when we replace $S^{[2]}$ with $S^{[n]}$. In §2 we show that

$$RF = \mathrm{id}_S \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2n+2].$$

Markman and Mehrotra [34] have announced a different proof of this using the Bridgeland-King-Reid–Haiman equivalence $D^b(S^{[n]}) \cong D^b([S^n/\mathfrak{S}_n])$; our proof is more geometric. To get an autoequivalence of $D^b(S^{[n]})$ from our functor F, we are obliged to generalize Huybrechts and Thomas's $\mathbb{P}$-objects [22]. In §3 we define $\mathbb{P}$-functors, give more examples of them, and show that they yield autoequivalences.

The behavior we are seeing seems to be about $S^{[n]}$ as a moduli space, not about Hilbert schemes à la Nakajima and Grojnowski [36, 17]: we do not get a family of $\mathbb{P}^n$-functors $D^b(S^{[m]}) \to D^b(S^{[m+n]})$, nor do we get anything if the surface S is not a K3. We make the following conjecture:

Conjecture. *Let $\mathcal{M}$ be a fine moduli space of stable sheaves on a K3 surface S, let $F : D^b(S) \to D^b(\mathcal{M})$ be the functor induced by the universal sheaf $\mathcal{U}$ on $S \times \mathcal{M}$, and let R be the right adjoint of F. Then*

$$RF = \mathrm{id}_S \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-\dim \mathcal{M} + 2]$$

and the monad structure $RFRF \to RF$ is like multiplication in the cohomology ring of projective space, so F determines an autoequivalence of $D^b(\mathcal{M})$.

Of course one should be willing to drop the hypothesis that $\mathcal{M}$ is fine and work with twisted sheaves. It does not seem feasible to prove this directly, as we do not know enough about the sheaves $\mathcal{U}|_{x \times \mathcal{M}}$ on $\mathcal{M}$, where $x \in S$, but it might be proved by deformation theory.

In §4 we give the following non-commutative example as evidence for the conjecture. Let X be a cubic 4-fold and $\mathcal{A} = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^\perp \subset D^b(X)$, which should be thought of as a non-commutative K3 surface: its Serre functor is a shift by 2 [27, Cor. 4.4] and its first Hochschild homology vanishes [26], but if X is very general then $\mathcal{A}$ lacks points and line bundles.

The variety Y of lines on X, which is a hyperkähler 4-fold [5], is a moduli space of sheaves in $\mathcal{A}$, and an appropriate functor $F : \mathcal{A} \to Y$ has $RF = \mathrm{id} \oplus [-2]$, hence is spherical. The associated autoequivalence of $D^b(Y)$ is new, as it sends the structure sheaf of a point to a complex of rank 2.

This example is related to the first one as follows. For certain "Pfaffian" cubic 4-folds X, Beauville and Donagi [5] constructed a K3 surface S of genus 8 with $S^{[2]} \cong Y$, and Kuznetsov [28] showed that $\mathcal{A} \cong D^b(S)$. One can check that via these isomorphisms, our functors $D^b(S) \to D^b(S^{[2]})$ and $\mathcal{A} \to D^b(Y)$ agree, up to a shift and a line bundle on both sides. Now the deformation space of $S^{[2]} \cong Y$ is 21-dimensional, of which 20 come from deforming the cubic (these are the directions in which the hyperkähler manifold remains projective) and a different 20 come from deforming the K3 surface. One wonders how our autoequivalence deforms to the whole moduli space of hyperkähler 4-folds. Markman shows that the sheaf (0.1) deforms to a *twisted* sheaf on the self-product of any manifold in the moduli space, which is interesting.

Finally we note that when X contains a plane, Voisin [44] constructed an associated K3 surface S of genus 2 which posesses a natural Brauer class α of order 2, Kuznetsov [28] showed that $\mathcal{A} \cong D^b(S, \alpha)$, where the latter is the derived category of α-twisted sheaves, and Macrì and Stellari [32] showed that Y is a moduli space of α-twisted sheaves on S. Our functor $F : \mathcal{A} \to Y$ agrees with the one induced by the universal sheaf.

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1 Spherical Functors

1.1 Definition

Let X be a smooth n-dimensional complex manifold, and recall that an object $\mathcal{E} \in D^b(X)$ is called *spherical* if $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong H^*(S^n, \mathbb{C})$, where S^n is the n-dimensional sphere, and $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$. The *twist* around $\mathcal{E}$ is the functor $T : D^b(X) \to D^b(X)$ sending an object $\mathcal{F}$ to the the cone on the evaluation map

$$\mathcal{E} \otimes \mathrm{RHom}(\mathcal{E}, \mathcal{F}) \to \mathcal{F}.$$

This definition is slighly sloppy, since cones are not functorial, but by now the remedy is well-known: one can work with a DG-enhancement or with Fourier–Mukai kernels. We prefer the latter, so what we really mean is that T is induced by the object

$$\operatorname{cone}(\mathcal{E}^* \boxtimes \mathcal{E} \to \mathcal{O}_\Delta) \in D^b(X \times X).$$

Seidel and Thomas [40] showed that T is an equivalence.

Now an object of $D^b(X)$ is the same as a functor $D^b(\text{point}) \to D^b(X)$, so following Rouquier [39] and Anno [1], we consider any exact functor $F : \mathcal{A} \to \mathcal{B}$ between triangulated categories, with left and right adjoints $L, R : \mathcal{B} \to \mathcal{A}$. We define the *twist* T to be the cone on the counit $FR \xrightarrow{\epsilon} 1$ of the adjunction, so there is an exact triangle

$$FR \xrightarrow{\epsilon} \mathrm{id}_\mathcal{B} \to T, \tag{1.1}$$

and the *cotwist* C to be the cone on the unit:

$$\mathrm{id}_\mathcal{A} \xrightarrow{\eta} RF \to C. \tag{1.2}$$

(Of course we need to be in a situation where these cones make sense; we will return to this point in a moment.) We say that F is *spherical* if C is an equivalence and $R \cong CL$.[1] If $\mathcal{A}$ and $\mathcal{B}$ have Serre functors $S_\mathcal{A}$ and $S_\mathcal{B}$ then the latter condition is equivalent to $S_\mathcal{B} FC \cong FS_\mathcal{A}$. When F is spherical, T is an equivalence.

Observe that if $\mathcal{E} \in D^b(X)$ is a spherical object and $F = \mathcal{E} \otimes - : D^b(\text{point}) \to D^b(X)$ then $R = \mathrm{RHom}(\mathcal{E}, -)$, so T is exactly the twist defined before, $RF = \mathrm{RHom}(\mathcal{E}, \mathcal{E}) = \mathrm{id} \oplus [-n]$, the cotwist $C = [-n]$, and the condition $S_X FC \cong FS_{\text{point}}$ is $\omega_X[n] \otimes \mathcal{E}[-n] \cong \mathcal{E}$.

Let us say a word about the cones (1.1) and (1.2). If $\mathcal{A}$ and $\mathcal{B}$ are derived categories of sheaves or twisted sheaves on smooth projective varieties or compact complex manifolds and F is induced by a Fourier–Mukai kernel then R, RF, and FR are induced by kernels as well, the unit and counit are induced by maps of kernels, and the standard compatibilities among units and counits hold at the level of kernels [7, Appendix]. The same is true if $\mathcal{A}$ and $\mathcal{B}$ are admissible subcategories of these, because the projection functors are induced by kernels [29].[2] It is also possible to do business with

[1] Rouquier requires the triangle (1.2) to be split. Both he and Anno require a certain natural map $R \to CL$ to be an isomorphism, but this is difficult to check in practice, and in our proof below we will see that any isomorphism $R \cong CL$ will do.

[2] Recall that a full subcategory $\mathcal{A} \subset D^b(X)$ is called *admissible* if the inclusion i has left and right adjoints i^l and i^r. The main examples are the image of a fully faithful Fourier–Mukai functor and the orthogonal to an exceptional collection. The projection functors are ii^l and ii^r.

derived categories of non-compact or singular varieties if one says "proper" or "perfect" at the right moments, or with schemes [2]. Rouquier's interest was in constructible sheaves.

1.2 Examples

Spherical functors unify the following special cases:

1. Spherical objects, as we have discussed. The main examples of these are a line bundle on a Calabi–Yau and the structure sheaf of -2-curve in a surface (e.g. $\mathbb{P}^1$ in its cotangent bundle). Another is the structure sheaf of $(-1,-1)$-curve in a 3-fold X, in which case the twist can also be described as doing Bondal and Orlov's flopping equivalence [6] twice: $D^b(X) \to D^b(X^+) \to D^b(X)$.

2. Horja's EZ-spherical objects, described in [20] or more carefully in [21, p. 186ff]. These should be seen as family versions of spherical objects. They are spherical functors of the form $F = i_* q^*$, where i is an embedding and q a smooth bundle as in the diagram

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "q"'] & X \\ Z. & \end{tikzcd}$$

   For example, we could take $q : E \to Z$ to be a $\mathbb{P}^1$-bundle, X the total space of the relative cotangent bundle, and $i : E \to X$ the zero section.

3. Toda's fat spherical objects [42]. These are spherical functors $F : D^b(\operatorname{Spec} A) \to D^b(X)$, where A is an Artinian local $\mathbb{C}$-algebra. His first example generalizes the Atiyah flop example above to $(0,-2)$-curves.

   Toda is able to simplify the hypothesis that the cotwist C is an equivalence as follows. Let $\mathcal{E}' \in D^b(\operatorname{Spec} A \times X)$ be the Fourier–Mukai kernel, $\pi : \operatorname{Spec} A \times X \to X$ the projection, $0 \in \operatorname{Spec} A$ the closed point, and $\mathcal{E} = \mathcal{E}'|_{0\times X}$. Then his condition $\operatorname{Ext}^*_X(\pi_*\mathcal{E}', \mathcal{E}) \cong H^*(S^n, \mathbb{C})$ is equivalent to $RF\mathcal{O}_0 \cong \mathcal{O}_0 \oplus \mathcal{O}_0[-n]$. Because $\mathcal{O}_0$ generates $D^b(\operatorname{Spec} A)$, this shows that $C = [-n]$.

To date, most authors using spherical functors have been interested in braid group representations. In Seidel and Thomas's original paper showed that the twists around the -2-curves in the exceptional divisor of the crepant

resolution of the A_n surface singularity satisfy the braid relations. Khovanov and Thomas [25] constructed EZ-spherical functors from the cotangent bundles of some partial flag varieties to that of a complete flag variety and showed that the associated twists give a representation of the braid group, which they enrich to a representation of the "braid cobordism" 2-category. Cautis and Kamnitzer [9] considered a similar example and enriched the structure in a different direction, getting representations of $\mathfrak{sl}_2$ and of other Lie algebras in later papers. Many other authors are also involved, including Rouquier and Anno; for a more complete history see [10].

Donovan [12] gave an example in which the cotwist is more interesting than just a shift or a line bundle.[3] He considered certain tautological vector bundles E_1 on $\mathbb{P}^{n-1}$ and E_2 on $\mathrm{Gr}(2, n)$ and constructed a spherical functor $D^b(E_1) \to D^b(E_2)$ whose cotwist is, up to a shift and a line bundle, the twist around a spherical object on E_1. This can be extended to a sequence of vector bundles E_k on $\mathrm{Gr}(k, n)$ and functors $D^b(E_k) \to D^b(E_{k+1})$ which he expects to be spherical, with the twist of each being the cotwist of the next, again up to a shift and a line bundle.

Our Hilbert scheme example differs from the braid group examples and Donovan's example in that the latter are all Horja twists or nearly so, whereas our Fourier–Mukai kernel on $S \times S^{[2]}$ is supported everywhere. To put it another way, if $x, y \in S$ then $F\mathcal{O}_x$ and $F\mathcal{O}_y$ are orthogonal, but not by virtue of having disjoint support.

Our cubic example is unique in that the domain $\mathcal{A}$ of the spherical functor is not the derived category of a variety.

To these substantial examples we add the following ones, which are silly in that the twist is obviously an equivalence.

4. Let $j : D \to X$ be the inclusion of a smooth divisor and take $F = j^*$, so $R = j_*$. Then $RF = j_* j^* = j_*\mathcal{O}_D \otimes -$, so by rotating the exact triangle
$$\mathcal{O}_X(-D) \to \mathcal{O}_X \to j_*\mathcal{O}_D$$
we find that $C = \mathcal{O}_X(-D)[1] \otimes -$, which is an equivalence. The condition $R = CL$ holds because $L = j_! = j_*(\omega_j[-1] \otimes -)$ and $\omega_j = \omega_D \otimes j^*\omega_X^* = j^*\mathcal{O}_X(D)$ by the adjunction formula. For the twist, note

[3]We mention that if a spherical functor $F : D^b(X) \to D^b(Y)$ between smooth compact spaces has its cotwist $C = \mathcal{L}[k] \otimes -$ for some line bundle $\mathcal{L}$ on X and $k \in \mathbb{Z}$, then the condition $S_Y FC \cong FS_X$ implies that $k = \dim X - \dim Y$. This can be seen by comparing the cohomology sheaves of the two kernels on $X \times Y$.

that there is an exact triangle of functors [21, Cor. 11.4]

$$(\mathcal{O}_D(-D)[1] \otimes -) \to j^* j_* \to \mathrm{id}_D,$$

so $T = \mathcal{O}_D(-D)[2] \otimes -$.

5. Again let $j : D \to X$, but now take $F = j_*$; by a similar computation, $C = \mathcal{O}_D(D)[-1] \otimes -$, $R = CL$, and $T = \mathcal{O}_X(D) \otimes -$. This is the example in Anno's paper [1]. It is an EZ-spherical twist, and can be seen as a family version of the fact that the skyscraper sheaf at a point in a curve is a spherical object.

   This example and the previous one reflect Logvinenko's observation [3] that F is spherical with cotwist C and twist T if and only if R is spherical with cotwist $T^{-1}[1]$ and twist $C^{-1}[1]$.

6. Let $p : \tilde{X} \to X$ be a double cover branched over a divisor $D \subset X$ and $F = p^*$, so $R = p_*$. Then

$$RF = p_* p^* = p_* \mathcal{O}_{\tilde{X}} \otimes - = (\mathcal{O}_X \oplus \mathcal{O}_X(-\tfrac{1}{2}D)) \otimes -,$$

   so $C = \mathcal{O}_X(-\frac{1}{2}D) \otimes -$ is an equivalence. The condition $R = CL$ holds because $L = p_! = p_*(\omega_p \otimes -)$ and $\omega_p = p^* \mathcal{O}_X(\frac{1}{2}D)$. For the twist, note that there is an exact triangle of functors

$$p^* \mathcal{O}_X(-\tfrac{1}{2}D) \otimes \tau^* \to p^* p_* \to \mathrm{id}_{\tilde{X}},$$

   where $\tau : \tilde{X} \to \tilde{X}$ exchanges the two sheets of the cover, so $T = p^* \mathcal{O}_X(-\frac{1}{2}D)[1] \otimes \tau^*$.

Seidel and Thomas prove several propositions on getting spherical objects from exceptional objects. Recall that an object $\mathcal{E} \in D^b(X)$ is called *exceptional* if $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) = H^*(\text{point}, \mathbb{C})$; the main examples are line bundles on Fano varieties and some homogeneous vector bundles. Note that $\mathcal{E}$ is exceptional if and only if the functor $\mathcal{E} \otimes - : D^b(\text{point}) \to D^b(X)$ is fully faithful, so here we relate spherical functors to fully faithful functors.

Proposition. *Let $\mathcal{B}$ be a triangulated category with Serre functor $S_{\mathcal{B}}$ and $F : \mathcal{B} \to \mathcal{C}$ a spherical functor with cotwist $C = S_{\mathcal{B}}[k]$ for some integer k. If $i : \mathcal{A} \to \mathcal{B}$ is fully faithful then $F' := Fi$ is spherical with cotwist $C' = S_{\mathcal{A}}[k]$.*

Proof. Let i^l and i^r be the adjoints of i. Recall that $\mathcal{A}$ inherits a Serre functor from $\mathcal{B}$ by the formula $S_{\mathcal{A}} = i^r S_{\mathcal{B}} i$. Let L and R be the adjoints of

F, so $L' = i^l L$ and $R' = i^r R$ are the adjoints of F'. The unit $\mathrm{id}_{\mathcal{A}} \to R'F'$ is the composition

$$\mathrm{id}_{\mathcal{A}} \to i^r i \to i^r RFi,$$

and the first arrow is an isomorphism, so we find that

$$C' = i^r C i = i^r S_{\mathcal{B}} i[k] = S_{\mathcal{A}}[k].$$

Moreover we have

$$R' = i^r R = i^r CL = i^r S_{\mathcal{B}} L[k] = S_{\mathcal{A}} i^l L[k] = C'L'. \qquad \square$$

From this and our silly examples above, we recover the following examples of Seidel and Thomas:

4′. Let $j : D \to X$ be the inclusion of an anticanonical hypersurface (that is, $\omega_X = \mathcal{O}_X(-D)$, so D is Calabi–Yau) and $\mathcal{E} \in D^b(X)$ an exceptional object; then $j^*\mathcal{E}$ is spherical. To spell things out, the set-up $\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{F} \mathcal{C}$ from the proposition is $D^b(\text{point}) \xrightarrow{\mathcal{E}\otimes -} D^b(X) \xrightarrow{j^*} D^b(D)$, and in Example 4 above we saw that j^* was spherical with cotwist $\mathcal{O}_X(-D)[1] \otimes - = S_X[-\dim D]$.

   For example, take a smooth quartic in $\mathbb{P}^3$ or a smooth quintic in $\mathbb{P}^4$ and let $\mathcal{E}$ be a line bundle, or the tangent bundle.

5′. Let $j : D \to X$ be a smooth hypersurface with $j^*\omega_X = \mathcal{O}_D$ and $\mathcal{E} \in D^b(D)$ an exceptional object; then $j_*\mathcal{E}$ is spherical. Now we are looking at $D^b(\text{point}) \xrightarrow{\mathcal{E}\otimes -} D^b(D) \xrightarrow{j_*} D^b(X)$, and in Example 5 we saw that j_* was spherical with cotwist $\omega_D \otimes j^*\omega_X[-1] \otimes - = S_D[-\dim X]$.

   For example, take a -2-curve in a surface and let $\mathcal{E}$ be a line bundle.

6′. Let $p : X \to \mathbb{P}^2$ be a double cover branched over a sextic, so X is a K3 surface, and $\mathcal{E} \in D^b(\mathbb{P}^2)$ an exceptional object; then $p^*\mathcal{E}$ is spherical. Now we are looking at $D^b(\text{point}) \xrightarrow{\mathcal{E}\otimes -} D^b(\mathbb{P}^2) \xrightarrow{p^*} D^b(X)$, and in Example 6 we saw that p^* was spherical with cotwist $\mathcal{O}_{\mathbb{P}^2}(-3) \otimes - = S_{\mathbb{P}^2}[-2]$.

1.3 Splitting of FRF

The following simple observation will be the key to describing the action of T on cohomology, proving that it is an equivalence, and constructing the $\mathbb{P}$-twist associated to a $\mathbb{P}$-functor in §3. While the unit $\mathrm{id}_{\mathcal{B}} \xrightarrow{\eta} RF$

is not typically split, the map $F \xrightarrow{F\eta} FRF$ is naturally split: we have a commutative triangle

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FRF \arrow[d, "\epsilon F"] \\ & F. \end{tikzcd}$$

In the down-to-earth case $F = \mathcal{E} \otimes - : D(\text{point}) \to D(X)$, we are saying that while there is no natural map $\mathrm{RHom}(\mathcal{E}, \mathcal{E}) \to \mathbb{C}$ splitting the map $\mathbb{C} \to \mathrm{RHom}(\mathcal{E}, \mathcal{E})$ which sends 1 to the identity (the trace map does not work when $\operatorname{rank} \mathcal{E} = 0$, for example), if we tensor with $\mathcal{E}$ then the map $\mathcal{E} \to \mathcal{E} \otimes \mathrm{RHom}(\mathcal{E}, \mathcal{E})$ is split by the evaluation map $\mathcal{E} \otimes \mathrm{RHom}(\mathcal{E}, \mathcal{E}) \to \mathcal{E}$.

Extend the commutative triangle above to

$$\begin{tikzcd} & TF[-1] \arrow[d] \arrow[dr] & \\ F \arrow[r, "F\eta"] \arrow[dr, equal] & FRF \arrow[r] \arrow[d, "\epsilon F"] & FC \\ & F, & \end{tikzcd}$$

where the row and column are exact. By the octahedral axiom, $TF[-1] \to FC$ is an isomorphism. Note that this is true for any F, spherical or not.

Expanding further on the splitting $FRF \cong F \oplus TF[-1] \cong F \oplus FC$, we note that the identity map $FRF \to FRF$ can be written as the sum of the following two compositions:

$$FRF \to FC \xleftarrow{\cong} TF[-1] \to FRF$$
$$FRF \xrightarrow{\epsilon F} F \xrightarrow{F\eta} FRF.$$

Similarly, we have isomorphisms

$$RT[-1] \xrightarrow{\cong} CR$$
$$FC^l \xrightarrow{\cong} T^l F[1]$$
$$C^l L \xrightarrow{\cong} LT^l[1],$$

where T^l and C^l are the left adjoints of T and C, and natural splittings

$$RFR \cong R \oplus RT[-1] \cong R \oplus CR$$
$$FLF \cong F \oplus FC^l \cong R \oplus T^l F[1]$$
$$LFL \cong L \oplus C^l L \cong L \oplus LT^l[1]$$

whose idempotents can be described as in the previous paragraph.

The reader may remark that it would have been more convenient to work with $T[-1]$ rather than T. We have not done so in order to be consistent with [40], and because we want spherical twists to act as reflections on cohomology, not as the opposite of reflections.

1.4 Action on Cohomology

If $\mathcal{E}$ is a spherical object, the twist T sends $\mathcal{E}$ to $\mathcal{E}[-n+1]$ and acts as the identity on $\mathcal{E}^\perp$. If n is even, the induced action on cohomology is a reflection, sending the Mukai vector $v(\mathcal{E}) \in H^*(X, \mathbb{Q})$ to $-v(\mathcal{E})$ and acting as the identity on its orthogonal $v(\mathcal{E})^\perp$ under the Mukai pairing.

For a spherical functor $F : \mathcal{A} \to \mathcal{B}$, this is generalized as follows. We replace $\mathcal{E}$ with

$$\operatorname{im} F = \{FA : A \in \mathcal{A}\}$$

and $\mathcal{E}^\perp$ with

$$\begin{aligned}(\operatorname{im} F)^\perp &= \{B \in \mathcal{B} : \operatorname{Hom}(FA, B) = 0 \text{ for all } A \in \mathcal{A}\} \\ &= \{B \in \mathcal{B} : RB = 0\} \\ &=: \ker R\end{aligned}$$

where we mean "ker" in the sense of elementary algebra, not as in Fourier–Mukai kernels. In the proof of Theorem 1 we will see that $\operatorname{im} F$ and $\ker R$ span $\mathcal{B}$, although they do not generate $\mathcal{B}$—that is, an object that is left and right orthogonal to $\operatorname{im} F$ and $\ker R$ is zero, although not every object can be gotten from $\operatorname{im} F$ and $\ker R$ by taking cones.

Now T acts as the identity on $\ker R$, for if $B \in \ker R$ then the first term in the exact triangle

$$FRB \to B \to TB$$

vanishes, so $TB = B$. It acts on $\operatorname{im} F$ thus: if $FA \in \operatorname{im} F$ then $TFA = FCA[1]$ by the previous subsection. In particular, if $C = [-n]$ then T acts on $\operatorname{im} F$ by $[-n+1]$.

For the action of T on cohomology, suppose that $C = [-n]$ with n even, that the triangle (1.2) is split, so $RF = \mathrm{id}_{\mathcal{A}} \oplus [-n]$, and that $\mathcal{A} = D^b(X)$ and $\mathcal{B} = D^b(Y)$ for smooth compact spaces X and Y. Let $F^h : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ be the map on cohomology induced by F, which is injective because $R^h F^h = (RF)^h$ is multiplication by 2. Then $T^h : H^*(Y, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ acts by -1 on $\operatorname{im}(F^h)$ and by the identity on its orthogonal $(\operatorname{im}(F^h))^\perp$ under the Mukai pairing.

We are now in a position to show that if S is a K3 surface and $F : D^b(S) \to D^b(S^{[2]})$ a spherical functor with cotwist $[-2]$, then the twist T_F does not come from any known spherical twist on S via Ploog's map $\varphi : \mathrm{Aut}(D^b(S)) \hookrightarrow \mathrm{Aut}(D^b(S^{[2]}))$. Ploog's map uses the Bridgeland-King-Reid–Haiman equivalence $D^b(S^{[2]}) \cong D^b([S^2/\mathbb{Z}_2])$, where the latter is the derived category of the quotient stack, or equivalently the $\mathbb{Z}_2$-equivariant derived category of S^2. Suppose that $\mathcal{E} \in D^b(S)$ is a spherical object and $\mathcal{F} \in \mathcal{E}^\perp$.[4] The spherical twist $T_\mathcal{E}$ shifts $\mathcal{E}$ by -1 and fixes $\mathcal{F}$. Consider the objects

$$\left.\begin{array}{c} \mathcal{E} \boxtimes \mathcal{E} \\ (\mathcal{E} \boxtimes \mathcal{F}) \oplus (\mathcal{F} \boxtimes \mathcal{E}) \\ \mathcal{F} \boxtimes \mathcal{F} \end{array}\right\} \in D^b([S^2/\mathbb{Z}_2]). \tag{1.3}$$

Then $\varphi(T_\mathcal{E})$ shifts the first by -2, shifts the second by -1, and fixes the third. On the other hand, T_F shifts $\operatorname{im} F$ by -1 and fixes $\ker R$,[5] and $\operatorname{im} F \cup \ker R$ is a spanning class. Now we need the following:

Proposition. *Suppose that X is a smooth variety, $A, B \in D^b(X)$, and T an autoequivalence of $D^b(X)$ with $TA = A[i]$ and $TB = B[j]$ for some $i \neq j \in \mathbb{Z}$. Then $A \perp B$ and $B \perp A$.*

Proof. For all $k, m \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Hom}(A, B[k]) &= \mathrm{Hom}(T^m A, T^m B[k]) \\ &= \mathrm{Hom}(A[mi], B[k+mj]) \\ &= \mathrm{Hom}(A, B[k + m(j-i)]), \end{aligned}$$

which vanishes for $|m| \gg 0$ because X is smooth. Similarly, $\mathrm{Hom}(B, A[k]) = 0$ for all k. ∎

Thus if T_F were $T_\mathcal{E}$ or a shift of it, one of the objects in (1.3) would be orthogonal to $\operatorname{im} F$ and $\ker R$, hence would be zero, which is a contradiction.

In the introduction we observed that for the functor $F : D^b(S) \to D^b(S^{[2]})$ induced by the universal ideal sheaf, the twist T_F is not generated by shifts, line bundles, automorphisms of $S^{[2]}$, or twists around $\mathbb{P}^2$-objects, because these all preserve rank (up to sign) while T_F sends the structure

[4] One does not know that such an $\mathcal{F}$ exists for an arbitrary $\mathcal{E}$, but one does in all known examples. If $\mathcal{E}$ is a line bundle, take $\mathcal{F} = \mathcal{E} \otimes \mathcal{I}_x^* \otimes \mathcal{I}_y$, where $x, y \in S$ are distinct points. If $\mathcal{E}$ is the structure sheaf of a -2-curve, take $\mathcal{F} = \mathcal{O}_x$ for some point x not on the curve. For a construction of $\mathcal{F}$ when $\mathcal{E}$ is an arbitrary stable vector bundle, see [37, Example 1.24].

[5] We cannot rule out the possibility that $\ker R = \emptyset$, but this does not affect the argument that follows.

sheaf of a point to a rank-2 sheaf shifted by 2. Another known autoequivalence of $D^b(S^{[2]})$ is the following EZ-spherical twist: Consider the divisor

$$D = \{\xi \in S^{[2]} : \operatorname{supp} \xi \text{ is a single point}\},$$

which is the exceptional divisor of the Hilbert–Chow morphism $S^{[2]} \to S^{(2)}$. It is a $\mathbb{P}^1$-bundle over S—the projectivization of the tangent bundle, in fact. Let $j : D \to S^{[2]}$ be the inclusion and $p : D \to S$ the projection. Then from the examples and proposition in §1.2 we easily check that $j_* p^* : D^b(S) \to D^b(S^{[2]})$ is spherical with cotwist $[-2]$. But EZ-spherical twists preserve rank as well: if $\mathcal{F} \in D^b(S^{[2]})$ then from the triangle

$$j_* p^* p_* j^! \mathcal{F} \to \mathcal{F} \to T_{j_* p^*} \mathcal{F}$$

we see that

$$\begin{aligned} \operatorname{rank} T_{j_* p^*} \mathcal{F} &= \operatorname{rank} \mathcal{F} - \operatorname{rank} j_* p^* p_* j^! \mathcal{F} \\ &= \operatorname{rank} \mathcal{F} - 0. \end{aligned}$$

We would like to show that T_F is not in the subgroup generated by these rank-preserving equivalences and the image of Ploog's map φ, but this is too difficult for the present work.

1.5 Proof of Equivalence

We conclude with an alternate proof that T is an equivalence, following Ploog [37].

Theorem 1 (Rouquier, Anno)**.** *If $F : \mathcal{A} \to \mathcal{B}$ is a spherical functor with $\mathcal{B}$ indecomposable*[6] *then the twist T is an equivalence.*

Proof. By [21, Prop. 1.49], we can show that T is fully faithful by producing a spanning class $\Omega \subset \mathcal{B}$, that is, a set of objects whose left and right orthogonals ${}^\perp\Omega$ and $\Omega^\perp$ are zero, and showing that the natural map

$$\operatorname{Hom}(B, B'[i]) \to \operatorname{Hom}(TB, TB'[i]) \tag{1.4}$$

is an isomorphism for all $B, B' \in \Omega$ and $i \in \mathbb{Z}$.

Take $\Omega = \operatorname{im} F \cup \ker R$. In the previous subsection we observed that $(\operatorname{im} F)^\perp = \ker R$, and similarly ${}^\perp(\operatorname{im} F) = \ker L$, but since $R \cong CL$ and C

[6] That is, $\mathcal{B}$ cannot be written as the union of two mutually orthogonal subcategories. For example, $D^b(X)$ is indecomposable if and only if X is connected.

is an equivalence we have $\ker R = \ker L$. Thus if $B \perp \Omega$ then $B \perp \operatorname{im} F$, so $B \in \ker L$, but also $B \perp \ker L$, so $B \perp B$, so $B = 0$, and similarly if $\Omega \perp B$ then $B = 0$, so Ω is a spanning class.

We check that (1.4) is an isomorphism in four cases. Since Ω is closed under shifts we need only consider $i = 0$. First, if $B, B' \in \ker R$ then $TB = B$ and $TB' = B'$, as we saw in the previous subsection, so $\operatorname{Hom}(TB, TB') = \operatorname{Hom}(B, B')$. Next, if $FA \in \operatorname{im} F$ and $B \in \ker R = \ker L$ then

$$\begin{aligned}\operatorname{Hom}(TFA, TB) &= \operatorname{Hom}(FCA[1], B) \\ &= \operatorname{Hom}(CA[1], RB) = 0 = \operatorname{Hom}(FA, B)\end{aligned}$$

$$\begin{aligned}\operatorname{Hom}(TB, TFA) &= \operatorname{Hom}(B, FCA[1]) \\ &= \operatorname{Hom}(LB, CA[1]) = 0 = \operatorname{Hom}(B, FA).\end{aligned}$$

Last, if $FA, FA' \in \operatorname{im} F$ then

$$\begin{aligned}\operatorname{Hom}(TFA, TFA') &= \operatorname{Hom}(T^l TFA, FA') = \operatorname{Hom}(T^l FCA[1], FA') \\ &= \operatorname{Hom}(FC^l CA, FA') = \operatorname{Hom}(FA, FA')\end{aligned}$$

where last step $C^l C = \mathrm{id}_{\mathcal{A}}$ is because C is an equivalence. But this is not quite enough—we need to show that the

$$T^l TF \xrightarrow{\epsilon F} F$$

is an isomorphism. The chain of equalities above suggests showing that it equals the composition

$$T^l TF \cong T^l FC[1] \cong FC^l C \xrightarrow{F\epsilon} F.$$

This is terribly boring, and we prove it as a separate lemma below; in fact they are the same up to a sign, which is good enough.

Now T is fully faithful, so by [21, Ex. 1.51] we can show that it is an equivalence by showing that $\ker T^l = 0$. If $B \in \ker T^l$ then $C^l LB = LT^l B[1] = 0$, but C^l is an equivalence, so $LB = 0$. Take left adjoints of (1.1) to get an exact triangle

$$T^l \to \mathrm{id}_{\mathcal{B}} \to FL,$$

from which we see that if $T^l B = 0$ then $B = 0$. $\square$

Lemma. *Let $F : \mathcal{A} \to \mathcal{B}$ be any Fourier–Mukai functor (not necessarily spherical), T and C the associated twist and cotwist, and T^l and C^l their left adjoints. Then the compositions*

$$T^l FC[1] \cong T^l TF \xrightarrow{\epsilon F} F$$
$$T^l FC[1] \cong FC^l C \xrightarrow{F\epsilon} F,$$

where the isomorphisms $\cong$ are the ones constructed in §1.3, are equal up to a sign.

Proof. First note that for any two functors $\Phi, \Psi : \mathcal{C} \to \mathcal{D}$, a natural transformation $\tau : \Phi \to \Psi$ determines a natural transformation $\tau^l : \Phi^l \to \Psi^l$ between their left adjoints, and the diagram

$$\begin{tikzcd}
\Psi^l \Phi \arrow[r, "\tau^l \Phi"] \arrow[d, "\Psi^l \tau"'] & \Phi^l \Phi \arrow[d, "\epsilon"] \\
\Psi^l \Psi \arrow[r, "\epsilon"'] & \mathrm{id}_{\mathbb{C}}
\end{tikzcd}$$

commutes. This is clear if $\mathcal{C}$ and $\mathcal{D}$ are the derived categories of smooth compact spaces, Φ and Ψ Fourier–Mukai functors, and τ induced by a map of kernels, because Φ^l and Ψ^l are induced by the dual kernels (tensored with the shift of a line bundle). But it is true in any category: τ^l is the composition $\Psi^l \xrightarrow{\Psi^l \eta} \Psi^l \Phi \Phi^l \xrightarrow{\Psi^l \tau \Phi^l} \Psi^l \Psi \Phi^l \xrightarrow{\epsilon \Phi^l} \Phi^l$, and one easily checks that the diagram above commutes.

Taking $\mathcal{C} = \mathcal{D} = \mathcal{B}$ and $\Phi \to \Psi$ to be $T[-1] \to FR$ or $RF \to C$, we get a commutative diagram

$$\begin{tikzcd}
 & FC^l RF \arrow[r] \arrow[d] & FC^l C \arrow[d] \arrow[ddr, "F\epsilon"] & \\
FLTF[-1] \arrow[r] \arrow[d] & FLFRF \arrow[r] \arrow[d] \arrow[drr, "F\epsilon"] \arrow[ddr, "\epsilon F"] & FLFC \arrow[d] & \\
T^l TF \arrow[r] \arrow[drr, "\epsilon F"] & T^l FRF[1] \arrow[r] & T^l FC[1] & F \\
 & & F &
\end{tikzcd}$$

in which the two vertical compositions and the two horizontal compositions are isomorphisms. The composition $FLFRF \to T^l FC[1]$ is an epimorphism, so it suffices to show that the compositions

$$FLFRF \to T^l FC[1] \overset{\cong}{\leftarrow} TT^l F \xrightarrow{\epsilon F} F \tag{1.5}$$

$$FLFRF \to T^l FC[1] \overset{\cong}{\leftarrow} FC^l C \xrightarrow{F\epsilon} F \tag{1.6}$$

are equal up to a sign. Rewrite (1.5) in the following steps:

$$FLFRF \to FLFC \to T^l FC[1] \overset{\cong}{\leftarrow} TT^l F \xrightarrow{\epsilon F} F$$

$$FLFRF \to FLFC \overset{\cong}{\leftarrow} FLTF[-1] \to TT^l F \xrightarrow{\epsilon F} F$$

$$FLFRF \to FLFC \overset{\cong}{\leftarrow} FLTF[-1] \to FLFRF \xrightarrow{\epsilon F} F.$$

In §1.3 we saw that the idempotent

$$FLFRF \to FLFC \overset{\cong}{\leftarrow} FLTF[-1] \to FLFRF$$

is the identity minus

$$FLFRF \xrightarrow{FL\epsilon F} FLF \xrightarrow{FLF\eta} FLFRF.$$

Moreover, the map $FLFRF \xrightarrow{\epsilon F} F$ can be factored as

$$FLFRF \xrightarrow{F\epsilon RF} FRF \xrightarrow{\epsilon F} F.$$

Compose these two to get

$$\begin{tikzcd}
 & & FLFRF \arrow[dr, "F\epsilon RF"] & & \\
FLFRF \arrow[r, "FL\epsilon F"] & FLF \arrow[ur, "FLF\eta"] \arrow[dr, "F\epsilon"'] & & FRF \arrow[r, "\epsilon F"] & F, \\
 & & F \arrow[ur, "F\eta"] \arrow[urr, equal, bend right] & &
\end{tikzcd}$$

that is, $FLFRF \xrightarrow{F\epsilon} F$. Thus (1.5) is $FLFRF \xrightarrow{\epsilon F - F\epsilon} F$, and similarly (1.6) is $FLFRF \xrightarrow{F\epsilon - \epsilon F} F$. $\square$

2 Hilbert Scheme Calculation

In this section we prove the following:

Theorem 2. *Let S be a complex K3 surface, $S^{[n]}$ the Hilbert scheme (or Douady space) of length-n subschemes of S, $Z = Z_n \subset S \times S^{[n]}$ the universal subscheme, $F : D^b(S) \to D^b(S^{[n]})$ be the functor induced by the ideal sheaf $\mathcal{I}_Z$, and R its right adjoint. Then*

$$RF = \mathrm{id}_S \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2n+2]$$

and the map

$$[-2] \oplus [-4] \oplus \cdots \oplus [-2n] \to \mathrm{id}_S \oplus [-2] \oplus \cdots \oplus [-2n+2]$$

given by

$$RF[-2] \hookrightarrow RFRF \xrightarrow{R\epsilon F} RF$$

is of the form

$$\begin{pmatrix} * & * & & & & \\ 1 & * & * & & & \\ & 1 & * & * & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & * & * \\ & & & & 1 & * \end{pmatrix}.$$

Presumably the monad structure $RFRF \xrightarrow{R\epsilon F} RF$ is like multiplication in $H^*(\mathbb{P}^{n-1}, \mathbb{C})$, but I have not managed to prove this. Note that the zeroes in the matrix above come for free, because

$$\mathrm{Hom}_{D^b(S)}(\mathcal{O}_\Delta[i], \mathcal{O}_\Delta[j]) = HH^{j-i}(S) = \begin{cases} \mathbb{C} & j-i=0 \\ \mathbb{C}^{22} & j-i=2 \\ \mathbb{C} & j-i=4 \\ 0 & \text{otherwise.} \end{cases}$$

In addition to F, we will consider the functors $F', F'' : D^b(S) \to D^b(S^{[n]})$ induced by $\mathcal{O}_{S\times S^{[n]}}$ and $\mathcal{O}_Z$ respectively, and their right adjoints R' and R''. We have exact triangles of functors

$$\begin{gathered} F \to F' \to F'' \\ R'' \to R' \to R. \end{gathered}$$

In §2.1 we give an exposition the "nested Hilbert scheme" which will be vital to our computations. In §§2.2–2.4 we compute $R'F'$, $R'F''$, $R''F'$, and $R''F''$, and enough information about the maps between them to determine RF through some long exact sequences. In §2.5 we prove the statement about the monad structure.

2.1 Nested Hilbert Schemes

The nested Hilbert scheme is

$$S^{[n-1,n]} = \{(\zeta, \xi) \in S^{[n-1]} \times S^{[n]} : \zeta \subset \xi\}.$$

Like $S^{[n]}$, it is smooth of dimension $2n$ [41]. We give a quick tour of its geometry, following Ellingsrud and Strømme [15]. This discussion is valid for any smooth surface.

For motivation, recall that $S^{[2]}$ has a very simple construction: let $\Delta \subset S \times S$ be the diagonal; then the involution of $S \times S$ lifts to $\mathrm{Bl}_\Delta(S \times S)$, fixing the exceptional divisor E, and the quotient is $S^{[2]}$. We summarize this in the diagram

$$\begin{tikzcd} E \subset & \mathrm{Bl}_\Delta(S \times S) \arrow[r, "g"] \arrow[d, "\gamma"'] & S^{[2]} \\ & S \times S. & \end{tikzcd}$$

The map $\pi_1\gamma \times g : \mathrm{Bl}_\Delta(S \times S) \to S \times S^{[2]}$ is an embedding, and its image is Z_2.

For $n > 2$, the picture will be

$$\begin{tikzcd} E \subset & S^{[n-1,n]} \arrow[r, "g"] \arrow[d, "\gamma = q \times f"'] & S^{[n]} \\ & S \times S^{[n-1]}, & \end{tikzcd}$$

where $f : S^{[n-1,n]} \to S^{[n-1]}$ and $g : S^{[n-1,n]} \to S^{[n]}$ are the obvious maps and $q : S^{[n-1,n]} \to S$ sends a pair $\zeta \subset \xi$ to the point where they differ, that is, where the kernel of $\mathcal{O}_\xi \to \mathcal{O}_\zeta$ is supported, which we will call $\xi \setminus \zeta$. (But note that there is no similar map $S^{[n-m,n]} \to S^{[m]}$ for $m > 1$, because $\ker(\mathcal{O}_\xi \to \mathcal{O}_\zeta)$ need not be a quotient of $\mathcal{O}_S$.)

Let $\phi = q \times g : S^{[n-1,n]} \to S \times S^{[n]}$. From the exact sequence

$$0 \to \mathcal{O}_{\xi\setminus\zeta} \to \mathcal{O}_\xi \to \mathcal{O}_\zeta \to 0$$

we see that the fiber of ϕ over (x,ξ) is $\mathbb{P}\operatorname{Hom}(\mathcal{O}_x,\mathcal{O}_\xi)^*$.[7] Thus the image of ϕ is Z_n, and ϕ is an isomorphism over the set of $(x,\xi)\in Z_n$ where the length of ξ at x is 1, so ϕ is a resolution of singularities. Since the fibers of ϕ are projective spaces,

$$\phi_*\mathcal{O}_{S^{[n-1,n]}} = \mathcal{O}_{Z_n},$$

so Z_n has rational singularities.

Next let $\gamma = q\times f : S^{[n-1,n]}\to S\times S^{[n-1]}$. From the exact sequence

$$0\to\mathcal{I}_\xi\to\mathcal{I}_\zeta\to\mathcal{O}_{\xi\setminus\zeta}\to 0$$

we see that the fiber of γ over (x,ζ) is $\mathbb{P}\operatorname{Hom}(\mathcal{I}_\zeta,\mathcal{O}_x)^* = \mathbb{P}(\mathcal{I}_\zeta|_x)$, so $S^{[n-1,n]}$ is isomorphic to the projectivization[8]

$$\mathbb{P}\mathcal{I}_{Z_{n-1}} = \operatorname{Proj}(\mathcal{O}_{S\times S^{[n-1]}}\oplus\mathcal{I}_{Z_{n-1}}\oplus\operatorname{Sym}^2\mathcal{I}_{Z_{n-1}}\oplus\cdots).$$

The blowup

$$\operatorname{Bl}_{Z_{n-1}}(S\times S^{[n-1]}) = \operatorname{Proj}(\mathcal{O}_{S\times S^{[n-1]}}\oplus\mathcal{I}_{Z_{n-1}}\oplus\mathcal{I}^2_{Z_{n-1}}\oplus\cdots)$$

naturally embeds into $\mathbb{P}\mathcal{I}_{Z_{n-1}}\cong S^{[n-1,n]}$, and since the latter is smooth, hence irreducible, the embedding is an isomorphism. Note that the rational map $g\circ\gamma^{-1}:\ S\times S^{[n-1]}\dashrightarrow S^{[n]}$ just sends a pair $(x,\zeta)\notin Z_{n-1}$ to $x\cup\zeta$.

Now Z_{n-1} is singular for $n>3$, and it is perhaps strange to blow up a smooth variety along a singular center and end up with a smooth variety. But γ behaves in many ways like a blowup along a smooth center. First,

$$\gamma_*\mathcal{O}_{S^{[n-1,n]}} = \mathcal{O}_{S\times S^{[n-1]}}$$

since the fibers of γ are projective spaces. Second, the exceptional divisor

$$E = \gamma^{-1}(Z_{n-1}) = \{(\zeta,\xi)\in S^{[n-1,n]} : (\xi\setminus\zeta)\in\zeta\}$$

is irreducible [15, §3]. Third,

Proposition. $\omega_{S^{[n-1,n]}} = \mathcal{O}(E)$.

[7] In this section only, we follow Grothendieck's convention that $\mathbb{P}$ is the projective space of 1-dimensional quotients. The reason will be clear in the next paragraph.

[8] For the reader uncomfortable with projectivizing sheaves that are not vector bundles, I recommend [14, pp. 103, 115, and 170–171]. Briefly, if $x\in X$ then the fiber of $\mathbb{P}\mathcal{I}_x$ over x is the projectivized tangent space, whereas the fiber of $\operatorname{Bl}_x(X)$ is the projectivized tangent cone.

Proof. Let Z'_{n-1} be the singular locus of Z_{n-1} and $E' = \gamma^{-1}(Z'_{n-1})$; then the claim is immediate on $S^{[n-1,n]} \setminus E'$, and since E' is a proper subset of the irreducible divisor E, it has codimension at least 2 in $S^{[n-1,n]}$, so the claim follows by Hartogs' theorem. □

Finally, although we will not need it, it is easy to check that $\gamma_* \mathcal{O}_E(E) = 0$.

We will conclude with the following fact, which we will need:

Proposition. *The map q is a submersion.*

Proof. This can be proved by working directly with the tangent spaces, but the proof is messy. Instead we give a quick transcendental proof. By Sard's theorem, q is a submersion over almost all $x \in S$. If $S = \mathbb{C}^2$, this implies that q is a submersion everywhere by translation. Now for any smooth surface S, let $(\zeta, \xi) \in S^{[n-1,n]}$ and let U be an analytic neighborhood of $\operatorname{supp} \xi$ isomorphic to an open set in $\mathbb{C}^2$, possibly disconnected. Then $U^{[n-1,n]}$ is a neighborhood of (ζ, ξ), and we have

$$\begin{tikzcd}
U^{[n-1,n]} \arrow[r, hook] \arrow[d, "q"'] & (\mathbb{C}^2)^{[n-1,n]} \arrow[d, "q"] \\
U \arrow[r, hook] & \mathbb{C}^2.
\end{tikzcd}$$

The horizontal maps are open immersions, and we have just seen that the right-hand q is a submersion. □

2.2 $R'F'$, $R'F''$, $R''F'$, and the maps between them

Recall that $F' : D^b(S) \to D^b(S^{[n]})$ is the functor induced by $\mathcal{O}_{S \times S^{[n]}}$ and R' its right adjoint, or more concretely,

$$\begin{aligned}
F'(-) &= \mathcal{O}_{S^{[n]}} \otimes_{\mathbb{C}} \mathrm{R}\Gamma(-) \\
R'(-) &= \mathcal{O}_S \otimes_{\mathbb{C}} \mathrm{R}\Gamma(-)[2].
\end{aligned}$$

Thus we immediately have $R'F'(-) = \mathcal{O}_S \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n]}}) \otimes \mathrm{R}\Gamma(-)[2]$, or equivalently,

$$R'F' = \mathcal{O}_{S \times S}[2] \oplus \mathcal{O}_{S \times S} \oplus \mathcal{O}_{S \times S}[-2] \oplus \cdots \oplus \mathcal{O}_{S \times S}[-2n+2].$$

(In what follows it will often be convenient to use the same name to refer to a functor and the kernel that induces it.)

Next, F'' is the functor induced by $\mathcal{O}_Z$; to simplify this, consider the diagram

$$\begin{tikzcd} S^{[n-1,n]} \arrow[r, "g"] \arrow[d, "q"'] & S^{[n]} \\ S & \end{tikzcd}$$

and recall that $(q \times g)_* \mathcal{O}_{S^{[n-1,n]}} = \mathcal{O}_Z$, so $F'' = g_* q^*$ and $R'' = q_* g^!$. Thus

$$\begin{aligned} R'F''(-) &= \mathcal{O}_S \otimes \mathrm{R}\Gamma(g_* q^* -)[2] \\ &= \mathcal{O}_S \otimes \mathrm{R}\Gamma(- \otimes q_* \mathcal{O}_{S^{[n-1,n]}})[2]. \end{aligned}$$

But factor q as

$$S^{[n-1,n]} \xrightarrow{q \times f} S \times S^{[n-1]} \xrightarrow{\pi_1} S$$

and recall that $(q \times f)_* \mathcal{O}_{S^{[n-1,n]}} = \mathcal{O}_{S \times S^{[n-1]}}$, so

$$\begin{aligned} q_* \mathcal{O}_{S^{[n-1,n]}} &= \mathcal{O}_S \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}}) \\ &= \mathcal{O}_S \oplus \mathcal{O}_S[-2] \oplus \cdots \oplus \mathcal{O}_S[-2n+2]. \end{aligned}$$

Thus

$$R'F'' = \mathcal{O}_{S \times S}[2] \oplus \mathcal{O}_{S \times S} \oplus \mathcal{O}_{S \times S}[-2] \oplus \cdots \oplus \mathcal{O}_{S \times S}[-2n+4].$$

Next I claim that the map $R'F' \to R'F''$ induces an isomorphism on $\mathcal{H}^i$ for $i < 2n-2$, so

$$R'F = \mathcal{O}_{S \times S}[-2n+2].$$

This amounts to the assertion that in the diagram

$$\begin{tikzcd} S^{[n-1,n]} \arrow[r, "q \times g"] \arrow[d, "q \times f"'] & S \times S^{[n]} \arrow[dd, "\pi_1'"] \\ S \times S^{[n-1]} \arrow[dr, "\pi_1"'] & \\ & S, \end{tikzcd} \tag{2.1}$$

the restriction map $\mathcal{O}_{S \times S^{[n]}} \to (q \times g)_* \mathcal{O}_{S^{[n-1,n]}}$ induces an isomorphism on $R^i \pi'_{1*}$ for $i < 2n$. We check this fiberwise. Since q is a submersion, its fibers

are smooth. Over a point $x \in S$, the fiber of (2.1) is

$$\begin{tikzcd} q^{-1}(x) \arrow[r, "g"] \arrow[d, "f"'] & S^{[n]} \arrow[dd] \\ S^{[n-1]} \arrow[dr] & \\ & x. \end{tikzcd}$$

Now we want to show that $g^* : H^i(\mathcal{O}_{S^{[n]}}) \to H^i(\mathcal{O}_{q^{-1}(x)})$ is an isomorphism for $i < 2n$. Let σ be a non-vanishing holomorphic 2-form on S, and σ_{n-1} and σ_n the induced holomorphic 2-forms on $S^{[n-1]}$ and $S^{[n]}$ constructed by Beauville [4, Prop. 5]. From his construction it is easy to check that on $S^{[n-1,n]}$ we have $g^*\sigma_n = q^*\sigma + f^*\sigma_{n-1}$. Thus the generator $\bar{\sigma}_n^j$ of $H^{2j}(\mathcal{O}_{S^{[n]}})$ maps to $f^*\bar{\sigma}_{n-1}^j \in H^{2j}(\mathcal{O}_{q^{-1}(x)})$. But since $f_*\mathcal{O}_{q^{-1}(x)} = \mathcal{O}_{S^{[n-1]}}$, the map $f^* : H^i(\mathcal{O}_{S^{[n-1]}}) \to H^i(\mathcal{O}_{q^{-1}(x)})$ is an isomorphism, so $f^*\bar{\sigma}_{n-1}^j$ generates $H^{2j}(\mathcal{O}_{q^{-1}(x)})$ for $j < n$, as desired.

By duality we have

$$R''F' = \mathcal{O}_{S\times S} \oplus \mathcal{O}_{S\times S}[-2] \oplus \cdots \oplus \mathcal{O}_{S\times S}[-2n+2],$$

and the map $R''F' \to R'F'$ induces an isomorphism on $\mathcal{H}^i$ for $i > -2$, so $RF' = \mathcal{O}_{S\times S}[2]$.

2.3 Main Calculation: $R''F''$

In this section we show that

$$\begin{aligned} R''F'' = \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[-2] \oplus \cdots \oplus \mathcal{O}_\Delta[-2n+4] \oplus \mathcal{O}_\Delta[-2n+2] \\ \oplus \mathcal{O}_{S\times S} \oplus \mathcal{O}_{S\times S}[-2] \oplus \cdots \oplus \mathcal{O}_{S\times S}[-2n+4]. \end{aligned} \tag{2.2}$$

The essential reason is this: we have $R''F'' = f_*g^!g_*f^*$, and we could compute $g^!g_*$ by base changing to

$$S^{[n-1,n]} \times_{S^{[n]}} S^{[n-1,n]} = \{(\zeta, \zeta', \xi) \in S^{[n]} \times S^{[n]} \times S^{[n-1]} : \zeta, \zeta' \subset \xi\},$$

which has two irreducible components: the diagonal $S^{[n-1,n]}$, and the rest, which is birational to $S \times S \times S^{[n-2]}$. These two components are responsible for the two lines of (2.2). We mention this now for fear that it will be obscured in the computation that follows.

Let π_{13} be the projection $S \times S^{[n]} \times S \to S \times S$, so

$$R''F'' = \pi_{13*}(\mathcal{O}_{Z\times S} \otimes \mathcal{O}^*_{S\times Z}[2]).$$

Then $\mathcal{O}_{Z\times S} \otimes \mathcal{O}^*_{S\times Z}$ is supported on $(Z\times S)\cap(S\times Z) = Z \times_{S^{[n]}} Z$. Consider the following partial resolution of singularities of $Z \times_{S^{[n]}} Z$:

$$\begin{aligned} X &:= Z \times_{S^{[n]}} S^{[n-1,n]} \\ &= \{(x,\zeta,\xi) \in S \times S^{[n-1,n]} : x \in \xi\}. \end{aligned}$$

In a moment we will describe the two irreducible components of X carefully, but for now let us observe that because Z is flat and finite over $S^{[n]}$, X flat and finite over $S^{[n-1,n]}$, and since the latter is smooth, X is Cohen–Macaulay, so every irreducible component of X is $2n$-dimensional.

The diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{\imath}"] \arrow[d, "\tilde{\phi}'"'] & S \times S^{[n-1,n]} \arrow[d, "\phi' := 1\times f\times q"] \\ Z \times S \arrow[r, "i"'] & S \times S^{[n]} \times S \end{tikzcd}$$

is also Cartesian, and we have

$$\begin{aligned} \mathcal{O}_{Z\times S} \otimes \mathcal{O}^*_{S\times Z}[2] &= (i_*\mathcal{O}_{Z\times S}) \otimes (\phi'_*\mathcal{O}_{S\times S^{[n-1,n]}})^*[2] \\ &= i_* i^* \phi'_*(\mathcal{O}_S \boxtimes \mathcal{O}(E)) \\ &= i_* \tilde{\phi}'_* \tilde{\imath}^*(\mathcal{O}_S \boxtimes \mathcal{O}(E)) \\ &= i_* \tilde{\phi}'_* \pi_2^* \mathcal{O}(E), \end{aligned}$$

where in the second line we have used the projection formula for i and Grothendieck duality for ϕ', and in the third we have used the base change criterion in Appendix A, and in the fourth, π_2 is the projection $X \to S^{[n-1,n]}$.

To see the two irreducible components of X, define maps

$$\begin{aligned} \delta : S^{[n-1,n]} &\to X & \qquad \epsilon : S^{[n-2,n-1,n]} &\to X \\ (\zeta,\xi) &\mapsto (\xi \setminus \zeta, \zeta, \xi) & (\eta,\zeta,\xi) &\mapsto (\zeta\setminus\eta, \zeta, \xi). \end{aligned}$$

Then $X = \operatorname{im}\delta \cup \operatorname{im}\epsilon$. In §2.A below we show that $S^{[n-2,n-1,n]}$, though not smooth [11], is irreducible. Since $X = \operatorname{im}\delta \cup \operatorname{im}\epsilon$, there is an exact sequence

$$0 \to \mathcal{I}_{(\operatorname{im}\delta \cap \operatorname{im}\epsilon)/\operatorname{im}\delta} \to \mathcal{O}_X \to \mathcal{O}_{\operatorname{im}\epsilon} \to 0$$

The map δ is a embedding, since $\pi_2\delta = \mathrm{id}$. The fiber of ϵ over (x, ζ, ξ) is $\mathbb{P}\,\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_\zeta)^*$, so $\epsilon_* \mathcal{O}_{S^{[n-2,n-1,n]}} = \mathcal{O}_{\mathrm{im}\,\epsilon}$. Finally, $\mathrm{im}\,\delta \cap \mathrm{im}\,\epsilon = \delta(E)$, so our exact sequence becomes

$$0 \to \delta_* \mathcal{O}(-E) \to \mathcal{O}_X \to \epsilon_* \mathcal{O}_{S^{[n-2,n-1,n]}} \to 0.$$

Tensor with $\pi_2^* \mathcal{O}(E)$ and use the projection formula to get

$$0 \to \delta_* \mathcal{O}_{S^{[n-1,n]}} \to \pi_2^* \mathcal{O}(E) \to \epsilon_* \epsilon^* \pi_2^* \mathcal{O}(E) \to 0.$$

We want to push this down to $S \times S$. For the first term, observe that the diagram

$$\begin{tikzcd}
S^{[n-1,n]} \arrow[r, "\delta"] \arrow[d, "q"'] & X \arrow[r, "i\tilde{\phi}'"] & S \times S^{[n]} \times S \arrow[d, "\pi_{13}"] \\
S \arrow[rr, "\Delta"'] & & S \times S
\end{tikzcd}$$

commutes, and we have seen that $q_* \mathcal{O}_{S^{[n-1,n]}} = \mathcal{O}_S \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}})$, so the first term becomes $\mathcal{O}_\Delta \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}})$. For the second term, observe that the composition

$$S^{[n-2,n-1,n]} \xrightarrow{\epsilon} X \xrightarrow{i\tilde{\phi}'} S \times S^{[n]} \times S \xrightarrow{\pi_{13}} S \times S$$

sends a point (η, ζ, ξ) to $(\zeta \setminus \eta, \xi \setminus \zeta)$, hence is the vertical composition in the diagram

$$\begin{tikzcd}
S^{[n-2,n-1,n]} \arrow[r, "\pi_2\epsilon"] \arrow[d] & S^{[n-1,n]} \arrow[d, "\gamma"] \\
S^{[n-2,n-1]} \times S \arrow[r] \arrow[d] & S^{[n-1]} \times S \\
S^{[n-2]} \times S \times S \arrow[d] & \\
S \times S. &
\end{tikzcd}$$

Using base change around the square (again by Appendix A) and the fact that $\gamma_* \mathcal{O}(E) = \mathcal{O}_{S^{[n-1]} \times S}$, we find that the second term becomes $\mathcal{O}_{S \times S} \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-2]}})$. Thus we have an exact triangle

$$\mathcal{O}_\Delta \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}}) \to R''F'' \to \mathcal{O}_{S \times S} \otimes \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-2]}}),$$

which must be split because $\mathrm{Ext}^i(\mathcal{O}_{S \times S}, \mathcal{O}_\Delta) = H^i(\mathcal{O}_S)$ vanishes when i is odd.

2.4 Cancellation

Now we will assemble what we know about $R'F'$, $R'F''$, $R''F'$, $R''F''$, and the maps between them to show that

$$RF = \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[-2] \oplus \cdots \oplus \mathcal{O}_\Delta[-2n+2].$$

We have a diagram of exact triangles

$$\begin{tikzcd} R''F' \arrow[r] \arrow[d] & R'F' \arrow[r] \arrow[d] & RF' \arrow[d] \\ R''F'' \arrow[r] & R'F'' \arrow[r] & RF''. \end{tikzcd}$$

Let us take cohomology sheaves of this to get a diagram of exact sequences

$$\begin{tikzcd} \mathcal{H}^i(R''F') \arrow[r] \arrow[d] & \mathcal{H}^i(R'F') \arrow[r] \arrow[d] & \mathcal{H}^i(RF') \arrow[r] \arrow[d] & \mathcal{H}^{i+1}(R''F') \arrow[r] \arrow[d] & \mathcal{H}^{i+1}(R'F') \arrow[d] \\ \mathcal{H}^i(R''F'') \arrow[r] & \mathcal{H}^i(R'F'') \arrow[r] & \mathcal{H}^i(RF'') \arrow[r] & \mathcal{H}^{i+1}(R''F'') \arrow[r] & \mathcal{H}^{i+1}(R'F'') \end{tikzcd}$$

for various i.

For $i = -2$ we have

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & ? \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd}$$

so

$$\mathcal{H}^{-2}(RF') = \mathcal{H}^{-2}(RF'') = \mathcal{O}$$

and the natural map between them is an isomorphism.

For $i = -1$ we have

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, equal] \arrow[d] & \mathcal{O} \arrow[d, equal] \\ 0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & \mathcal{O} \oplus \mathcal{O}_\Delta \arrow[r] & \mathcal{O}. \end{tikzcd}$$

Since the right-hand square is commutative, the map $\mathcal{O} \oplus \mathcal{O}_\Delta \to \mathcal{O}$ is split, so its kernel is $\mathcal{O}_\Delta$, so

$$\mathcal{H}^{-1}(RF') = 0 \qquad \mathcal{H}^{-1}(RF'') = \mathcal{O}_\Delta.$$

For $i = 0$ we have

$$\begin{tikzcd}
& \mathcal{O} \arrow[r, equal] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & ? \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& \mathcal{O} \oplus \mathcal{O}_\Delta \arrow[r] & \mathcal{O} \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{O} \oplus \mathcal{O}_\Delta \to \mathcal{O}$ is surjective, we get

$$\mathcal{H}^0(RF') = \mathcal{H}^0(RF'') = 0.$$

For $1 \leq i \leq 2n-4$ we get the same result as $i = -1$ and $i = 0$ over and over.

For $i = 2n-3$ we have

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, equal] \arrow[d] & \mathcal{O} \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & \mathcal{O}_\Delta \arrow[r] & 0,
\end{tikzcd}$$

so

$$\mathcal{H}^{2n-3}(RF') = 0 \qquad \mathcal{H}^{2n-3}(RF'') = \mathcal{O}_\Delta.$$

For $i = 2n-2$ we have

$$\begin{tikzcd}
\mathcal{O} \arrow[r, equal] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{O}_\Delta \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd}$$

so

$$\mathcal{H}^{2n-2}(RF') = \mathcal{H}^{2n-2}(RF'') = 0.$$

Now we take cohomology sheaves of the exact triangle

$$RF \to RF' \to RF''$$

to get a long exact sequence

$$\begin{array}{cclcccc}
0 & \to & \mathcal{H}^{-2}(RF) & \to & \mathcal{O} & = & \mathcal{O} \\
 & \to & \mathcal{H}^{-1}(RF) & \to & 0 & \to & \mathcal{O}_\Delta \\
 & \to & \mathcal{H}^{0}(RF) & \to & 0 & \cdots & 0 \\
 & \to & \mathcal{H}^{2n-3}(RF) & \to & 0 & \to & \mathcal{O}_\Delta \\
 & \to & \mathcal{H}^{2n-2}(RF) & \to & 0 & \to & 0
\end{array}$$

which gives

$$\mathcal{H}^i(RF) = \begin{cases} \mathcal{O}_\Delta & i = 0, 2, \dots, 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus RF has a filtration whose associated graded object is $\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[-2] \oplus \cdots \oplus \mathcal{O}_\Delta[-2n+2]$, but since $\operatorname{Ext}^k_{S\times S}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = HH^k(S)$ vanishes when k is odd, the filtration splits.

2.5 Monad Structure

Now we know that

$$RF = \mathrm{id}_S \oplus [-2] \oplus \cdots \oplus [-2n+2],$$

and we consider the monad structure $RFRF \xrightarrow{R\epsilon F} RF$. Presumably it is like multiplication in $H^*(\mathbb{P}^{n-1}, \mathbb{C})$, but we will only prove the weaker statement that the composition

$$RF[-2] \hookrightarrow RFRF \xrightarrow{R\epsilon F} RF$$

induces an isomorphism on $\mathcal{H}^i$ for $2 \le i \le 2n-2$, which is sufficient for our purposes in §3.

Again let q and g be as in

$$\begin{tikzcd} S^{[n-1,n]} \arrow[r, "g"] \arrow[d, "q"'] & S^{[n]} \\ S, & \end{tikzcd}$$

and consider the natural map $q_* q^* \xrightarrow{q_*\eta q^*} q_* g^! g_* q^* = R''F''$. This is a map of monads for formal reasons. Factor q as

$$S^{[n-1,n]} \xrightarrow{q\times f} S \times S^{[n-1]} \xrightarrow{\pi_1} S,$$

and recall that $(q \times f)_* \mathcal{O}_{S^{[n-1,n]}} = \mathcal{O}_{S\times S^{[n-1]}}$, so $(q\times f)_*(q\times f)^*$ is the identity, so we have

$$q_* q^* = \pi_{1*}\pi_1^* = - \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}})$$

as monads, where the monad structure in the last term is given by multiplication in the graded ring $\mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}})$.

To get a map $q_*q^* \to RF$, consider the diagram

$$\begin{tikzcd}
 & q_*q^* \arrow[d, dotted, "a"'] \arrow[dr] \arrow[ddr] & & \\
 & {} & R''F'' \arrow[d] & \\
R'F \arrow[r] & RF \arrow[r] & R''F[1] \arrow[r] & R'F[1]
\end{tikzcd}$$

in which the bottom row is exact. Applying $\mathrm{Hom}(q_*q^*, -)$ to the bottom row we get an exact sequence

$$\begin{aligned}
&\mathrm{Hom}(q_*q^*, R'F) \to \mathrm{Hom}(q_*q^*, RF) \\
&\qquad\qquad\qquad \to \mathrm{Hom}(q_*q^*, R''F[1]) \to \mathrm{Hom}(q_*, q^*R'F[1]),
\end{aligned}$$

where we emphasize that we mean Hom between Fourier–Mukai kernels on $S \times S$. But

$$\begin{aligned}
&\mathrm{Hom}(q_*q^*, R'F[1]) \\
&= \mathrm{Hom}_{S\times S}(\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[-2] \oplus \cdots \oplus \mathcal{O}_\Delta[-2n+2], \mathcal{O}_{S\times S}[-2n+3]) \\
&= \mathrm{Ext}^{-2n+3}_{S\times S}(\mathcal{O}_\Delta, \mathcal{O}_{S\times S}) \oplus \mathrm{Ext}^{-2n+5}_{S\times S}(\mathcal{O}_\Delta) \oplus \cdots \oplus \mathrm{Ext}^1_{S\times S}(\mathcal{O}_\Delta, \mathcal{O}_{S\times S}),
\end{aligned}$$

of which the negative Exts are necessarily zero and

$$\mathrm{Ext}^1_{S\times S}(\mathcal{O}_\Delta, \mathcal{O}_{S\times S}) = H^3(\mathcal{O}_\Delta)^* = H^3(\mathcal{O}_S)^* = 0$$

as well, and similarly $\mathrm{Hom}(f_*f^*, R'F) = 0$. Thus there is a unique lift a as in the diagram. From the diagram chase in the previous subsection we see that a induces an isomorphism on cohomology sheaves, hence is an isomorphism, but unfortunately we do not know that it is a monad map.

Now consider the functor

$$\begin{aligned}
\Phi &= \mathrm{id}_S \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n]}}) \\
&= \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[-2] \oplus \cdots \oplus \mathcal{O}_\Delta[-2n]
\end{aligned}$$

For formal reasons which we discuss below, there are monad maps b and c

as in the diagram

$$\begin{tikzcd}
\Phi \arrow[r, "c"] \arrow[ddr, "b"'] & q_*q^* \arrow[dr] \arrow[dd, dotted, "a"] \arrow[ddl, phantom, "?"] & & \\
& & R''F'' \arrow[d] & \\
R'F \arrow[r] & RF \arrow[r] & R''F[1] \arrow[r] & R'F[1]
\end{tikzcd}$$

which make the outer pentagon commute, although not necessarily the triangle marked ?: we do not know that $b = a \circ c$. But after composing with $RF \to R''F[1]$ they become equal, so they can differ only by a map $\Phi \to R'F \to RF$, and since $R'F = \mathcal{O}_\Delta[-2n+2]$ and there are no maps $\mathcal{O}_\Delta[i] \to \mathcal{O}_\Delta[-2n+2]$ for $i > -2n$, we find that b and $a \circ c$ must agree except perhaps on the last component $\mathcal{O}_\Delta[-2n]$ of Φ.

Below we will produce a map $d : q_*q^* \to \Phi$ splitting c. Observe that $a = b \circ d$, since after composing with $RF \to R''F[1]$ they become equal and $\operatorname{Hom}(q_*q^*, R'F) = 0$. Thus $d \circ a^{-1}$ splits b.

Let $2 \le k \le 2n-4$, and let $i : [-2] \hookrightarrow RF$ and $j : [-k] \hookrightarrow RF$ be the inclusions. We want to show that the map

$$[-k][-2] \xrightarrow{ij} RFRF \xrightarrow{R\epsilon F} RF$$

induces an isomorphism on $\mathcal{H}^{i+2}$. Let $\tilde{\imath} = d \circ a^{-1} \circ i$ and $\tilde{\jmath} = d \circ a^{-1} \circ j$. Then in the diagram

$$\begin{tikzcd}
& \Phi\Phi \arrow[r, "\mu"] \arrow[d, "bb"] & \Phi \arrow[d, "b"] \\
{[-k][-2]} \arrow[ur, "\tilde{\imath}\tilde{\jmath}"] \arrow[r, "ij"'] & RFRF \arrow[r, "R\epsilon F"'] & RF
\end{tikzcd}$$

we have $R\epsilon F \circ ij = b \circ \mu \circ \tilde{\imath}\tilde{\jmath}$. But because $-k-2 > -2n$, the map $\mu \circ \tilde{\imath}\tilde{\jmath}$ does not map to the last component $\mathcal{O}_\Delta[-2n]$ of Φ, so $b \circ \mu \circ \tilde{\imath}\tilde{\jmath} = a \circ c \circ \mu \circ \tilde{\imath}\tilde{\jmath}$. Thus we consider the diagram

$$\begin{tikzcd}
& \Phi\Phi \arrow[r, "\mu"] \arrow[d, "cc"] & \Phi \arrow[d, "c"] \\
{[-k][-2]} \arrow[ur, "\tilde{\imath}\tilde{\jmath}"] \arrow[r, "(a^{-1}\circ i)(a^{-1}\circ j)"'] & q_*q^*q_*q^* \arrow[r, "q_*\epsilon q^*"'] & q_*q^*.
\end{tikzcd}$$

Knowing the monad structure on q_*q^*, we know that $(q_*\epsilon q^*)\circ(a^{-1}\circ i)(a^{-1}\circ j)$ induces an isomorphism on $\mathcal{H}^{i+2}$, and a induces an isomorphism on all cohomology sheaves, so their composition, which equals $R\epsilon F \circ ij$, induces an isomorphism on $\mathcal{H}^{i+2}$, as desired.

It remains to produce the maps b and c, and the splitting d.

A class $\tau \in HH^i(S^{[n]})$, that is, a natural transformation $\tau : \mathrm{id}_{S^{[n]}} \to [i]$, determines a natural transformation $[-i] \to RF$, namely

$$[-i] \xrightarrow{\eta[-i]} RF[-i] \xrightarrow{R\tau F} RF. \tag{2.3}$$

In this way we get a natural transformation

$$\bigoplus HH^i(S^{[n]}) \otimes_{\mathbb{C}} [-i] \to RF.$$

One easily checks that this is a monad map, and that the diagram

$$\begin{tikzcd} \bigoplus HH^i(S^{[n]}) \otimes_{\mathbb{C}} [-i] \arrow[r] \arrow[d] & R''F'' \arrow[d] \\ RF \arrow[r] & R''F[1] \end{tikzcd}$$

commutes. This is purely formal—it has nothing to do with the details of F and F''.

Restricting to the subring $\mathrm{Ext}^*(\mathcal{O}_{S^{[n]}}, \mathcal{O}_{S^{[n]}}) \subset HH^*(S^{[n]})$, we find that the natural transformation (2.3) has a particularly simple description: given a map $\tau : \mathcal{O}_{S^{[n]}} \to \mathcal{O}_{S^{[n]}}[i]$, tensor with with $F(-)$ to get

$$\mathcal{O}_{S^{[n]}}[-i] \otimes F(-) \xrightarrow{\tau\otimes 1} \mathcal{O}_{S^{[n]}} \otimes F(-),$$

apply R, and precompose with the map $[-i] \xrightarrow{\eta[-i]} RF[i]$.

From this description we can see that the map

$$\mathrm{id}_S \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n]}}) \to R''F'' = q_* g^! g_* q^*$$

factors through q_*q^*, as follows. Given $\tau : \mathcal{O}_{S^{[n]}} \to \mathcal{O}_{S^{[n]}}[i]$, we have

$$\mathcal{O}_{S^{[n]}}[-i] \otimes g_*(-) \xrightarrow{\tau\otimes 1} \mathcal{O}_{S^{[n]}} \otimes g_*(-).$$

Apply $g^!$ to this and recall that $g^!(- \otimes -) = g^*(-) \otimes g^!(-)$, so we get

$$\mathcal{O}_{S^{[n-1,n]}}[-i] \otimes g^! g_*(-) \xrightarrow{g^*\tau\otimes 1} \mathcal{O}_{S^{[n-1,n]}} \otimes g^! g_*(-).$$

This yields the desired factorization

$$\mathrm{id}_S \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n]}}) \to q_* q^* \to R'' F''.$$

We can describe the map

$$\mathrm{id}_S \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n]}}) \xrightarrow{c} q_* q^*$$

quite explicitly. As at the end of §2.2, let σ be a holomorphic symplectic form on S and σ_{n-1} and σ_n the induced forms $S^{[n-1]}$ and $S^{[n]}$. Then $\bar{\sigma}_n^i$ generates $H^{2i}(\mathcal{O}_{S^{[n]}}) = \mathrm{Ext}^{2i}(\mathcal{O}_{S^{[n]}}, \mathcal{O}_{S^{[n]}})$, so we let $\mathcal{F} \in D^b(S)$ and consider

$$\mathcal{O}_{S^{[n-1,n]}}[-2i] \otimes q^* \mathcal{F} \xrightarrow{g^* \bar{\sigma}^i \otimes 1} \mathcal{O}_{S^{[n-1,n]}} \otimes q^* \mathcal{F}.$$

Now $g^* \sigma_n = f^* \sigma_{n-1} + q^* \sigma$, so

$$g^* \bar{\sigma}_n^i = f^* \bar{\sigma}_{n-1}^i + (q^* \bar{\sigma} \wedge f^* \bar{\sigma}_{n-1}^{i-1}),$$

that is, our map $\mathcal{O}_{S^{[n-1,n]}}[-i] \to \mathcal{O}_{S^{[n-1,n]}}$ is pulled back from a map $\mathcal{O}_{S \times S^{[n-1]}}[-i] \to \mathcal{O}_{S \times S^{[n-1]}}$ via $q \times f$. First consider $f^* \bar{\sigma}_{n-1}^i$: take

$$\mathcal{O}_{S^{[n-1,n]}}[-2i] \otimes q^* \mathcal{F} \xrightarrow{f^* \bar{\sigma}_{n-1}^i \otimes 1} \mathcal{O}_{S^{[n-1,n]}} \otimes q^* \mathcal{F},$$

apply $(q \times f)_*$ to get

$$\mathcal{O}_{S \times S^{[n-1]}}[-2i] \otimes \pi_1^* \mathcal{F} \xrightarrow{\pi_2^* \bar{\sigma}_{n-1}^i \otimes 1} \mathcal{O}_{S \times S^{[n-1]}} \otimes \pi_1^* \mathcal{F},$$

apply π_{1*} to get

$$\mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}})[-2i] \otimes \mathcal{F} \xrightarrow{\bar{\sigma}_{n-1}^i \otimes 1} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}}) \otimes \mathcal{F},$$

and precompose with the unit to get a map

$$\mathcal{F}[-2i] \to \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}}) \otimes \mathcal{F} = \mathcal{F} \oplus \mathcal{F}[-2] \oplus \cdots \oplus \mathcal{F}[-2n+2]$$

which is the identity on the $[-2i]$ piece and zero otherwise. On the other hand, $q^* \bar{\sigma} \wedge f^* \bar{\sigma}_{n-1}^{i-1}$ yields the map

$$\mathcal{F}[-2i] \to \mathrm{R}\Gamma(\mathcal{O}_{S^{[n-1]}}) \otimes \mathcal{F} = \mathcal{F} \oplus \mathcal{F}[-2] \oplus \cdots \oplus \mathcal{F}[-2n+2]$$

which is $\bar{\sigma} : \mathcal{F}[-2i] \to \mathcal{F}[-2i+2]$ and zero otherwise. Thus the map

$$\mathrm{id}_S \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{O}_{S^{[n]}}) \xrightarrow{c} q_* q^*,$$

when written in components

$$\mathrm{id}_S \oplus [-2] \oplus \cdots \oplus [-2n+2] \oplus [-2n] \to \mathrm{id}_S \oplus [-2] \oplus \cdots \oplus [-2n+2]$$

looks like

$$c = \begin{pmatrix} 1 & \bar{\sigma} & & & \\ & 1 & \bar{\sigma} & & \\ & & \ddots & \ddots & \\ & & & 1 & \bar{\sigma} \end{pmatrix},$$

and one easily writes down an $n \times (n-1)$-matrix d with $d \circ c = 1$.

2.A Appendix: Irreducibility of $S^{[n-2,n-1,n]}$

In this subsection we will show that if S is a smooth surface then the nested Hilbert scheme $S^{[n-2,n-1,n]}$ is irreducible, which we needed to know in §2.3. Note that not all nested Hilbert schemes are irreducible: for $n \gg 0$, $S^{[1,2,\ldots,n]}$ has components whose dimension is greater than the expected $2n$ [18].

For cleanliness we prefer to work with $S^{[n-1,n,n+1]}$. Recall that $S^{[n]}$ and $S^{[n-1,n]}$ are smooth of dimension $2n$ for all n. Write $S^{[n-1,n,n+1]}$ as the intersection

$$(S^{[n-1,n]} \times S^{[n+1]}) \cap (S^{[n-1]} \times S^{[n,n+1]}) \subset S^{[n-1]} \times S^{[n]} \times S^{[n+1]};$$

then because the ambient space is smooth, every component of the intersection has at least the expected dimension

$$\begin{aligned}[2n + (2n+2)] + [(2n-2) + (2n+2)] - [(2n-2) + 2n + (2n+2)]& \\ = 2n+2.&\end{aligned}$$

Consider the fiber square

$$\begin{tikzcd} S^{[n-1,n,n+1]} \arrow[r, "\tilde{g}"] \arrow[d] & S^{[n,n+1]} \arrow[d, "f"] \\ S^{[n-1,n]} \arrow[r, "g"'] & S^{[n]}. \end{tikzcd}$$

Let $U = S^{[n,n+1]} \setminus E$, where E is the exceptional divisor. Then the fibers of $f|_U$ are the fibers of $(S^{[n]} \times S) \setminus Z \to S^{[n]}$, which are irreducible surfaces (S minus finitely many points), so $\tilde{g}^{-1}(U) = S^{[n-1,n]} \times_{S^{[n]}} U$ is irreducible. Thus it is enough to show that $\dim \tilde{g}^{-1}(E) < 2n+2$.

The maps $g : S^{[n-1,n]} \to S^{[n]}$ and $f|_E : E \to S^{[n]}$ factor through Z. Ellingsrud and Strømme [15, §3] partition Z into locally closed subsets

$$W_i = \{(\zeta, x) \in Z : \dim(\mathcal{I}_\zeta|_x) = i\} \qquad\qquad i \geq 2$$

and show that the fiber of $E \to Z$ over W_i is $\mathbb{P}^{i-1}$, the fiber of $S^{[n-1,n]} \to Z$ over W_i is $\mathbb{P}^{i-2}$, and $\dim W_i \leq 2n + 4 - 2i$. Now $\tilde{g}^{-1}(E) = S^{[n-1,n]} \times_{S^{[n]}} E$ maps to $Z \times_{S^{[n]}} Z$, which we partition into

$$W_i \times_{S^{[n]}} W_j = \{(w, w') \in W_i \times W_j : \pi(w) = \pi(w')\},$$

where $\pi : Z \to S^{[n]}$. Since $Z \times_{S^{[n]}} Z \to S^{[n]}$ is finite, the dimension of $W_i \times_{S^{[n]}} W_j$ is the same as that of its image $\pi(W_i) \cap \pi(W_j)$, which is at most $2n + 4 - 2\max\{i, j\}$. Thus the preimage of $W_i \times_{S^{[n]}} W_j$ in $S^{[n-1,n]} \times_{S^{[n]}} E$ has dimension at most

$$(2n + 4 - 2\max\{i, j\}) + (i - 2) + (j - 1) \leq 2n + 1,$$

which gives the desired result.

3 $\mathbb{P}$-Functors

3.1 Definition

In view of the previous section, we need to define $\mathbb{P}$-functors, generalizing the $\mathbb{P}$-objects of Huybrechts and Thomas [22]. Let X be a smooth $2n$-dimensional complex manifold, and recall that an object $\mathcal{E} \in D^b(X)$ is called a *$\mathbb{P}^n$-object* if $\operatorname{Ext}^*(\mathcal{E}, \mathcal{E}) \cong H^*(\mathbb{P}^n, \mathbb{C})$ as algebras, and $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$. The first example is a line bundle on a hyperkähler manifold. The second is the structure sheaf of a $\mathbb{P}^n$ in a hyperkähler manifold, for example $\mathbb{P}^n$ sitting in the total space of its cotangent bundle, or if S is a K3 surface containing a rational curve $C \cong \mathbb{P}^1$ then $C^{[n]} \cong \mathbb{P}^n \subset S^{[n]}$.

We say that a functor $F : \mathcal{A} \to \mathcal{B}$ with left and right adjoints L and R is a *$\mathbb{P}^n$-functor* if the following conditions are satisfied:

a. There is an autoequivalence H of $\mathcal{A}$ such that

$$RF \cong \mathrm{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n. \tag{3.1}$$

b. The map

$$HRF \hookrightarrow RFRF \xrightarrow{R\epsilon F} RF,$$

when written in components

$$H \oplus H^2 \oplus \cdots \oplus H^n \oplus H^{n+1} \to \mathrm{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n,$$

is of the form

$$\begin{pmatrix} * & * & \cdots & * & * \\ 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix}.$$

(This models the fact that the map $\mathbb{C}[h]/h^{n+1} \xrightarrow{h\cdot} \mathbb{C}/h^{n+1}$ has kernel $\mathbb{C} \cdot h^n$ and cokernel $\mathbb{C} \cdot 1$.)

c. $R \cong H^n L$. If $\mathcal{A}$ and $\mathcal{B}$ have Serre functors, this is equivalent to $S_{\mathcal{B}} F H^n \cong F S_{\mathcal{A}}$.

As in §1 we need our functors to be induced by Fourier–Mukai kernels, and the isomorphism (3.1) to be induced by a map of kernels.

Cautis [8] has recently made a similar definition; he considers only $H = [-2]$, but we will see interesting examples with $H = [-1]$ as well. Of course one would like to find examples with more exciting H.

After giving examples of $\mathbb{P}$-functors, we construct the $\mathbb{P}$-twist associated to a $\mathbb{P}$-functor. There is some work to do beyond simply quoting Huybrechts and Thomas, because in taking a certain double cone we face a choice which does not arise for them. But thinking carefully about this choice will allow us to simplify their proof that the $\mathbb{P}$-twist is an equivalence.

3.2 Examples

1. If $\mathcal{E} \in D^b(X)$ is a $\mathbb{P}^n$-object then $F = \mathcal{E} \otimes - : D^b(\text{point}) \to D^b(X)$ is a $\mathbb{P}^n$-functor with $H = [-2]$.

2. The functor $F : D^b(S) \to D^b(S^{[n]})$ in the previous section is a $\mathbb{P}^{n-1}$-functor with $H = [-2]$. The condition $S_{S^{[n]}} F H^{n-1} \cong F S_S$ is satisfied because $S_{S^{[n]}} = [2n]$ and $S_S = [2]$.

3. A split spherical functor $F : \mathcal{A} \to \mathcal{B}$, that is, one where the exact triangle
$$\mathrm{id}_{\mathcal{A}} \xrightarrow{\eta} RF \to C$$
is split, so $RF \cong \mathrm{id}_{\mathcal{A}} \oplus C$, is a $\mathbb{P}^1$-functor with $H = C$. The $\mathbb{P}^1$-twist which we construct below is the square of the spherical twist, just as in [22, Prop. 2.9].

4. Let $q : E \to Z$ be a $\mathbb{P}^n$-bundle and $i : E \to \Omega^1_q$ the zero section of the relative cotangent bundle. Then $F := i_* q^*$ is a $\mathbb{P}^n$-functor with $H = [-2]$, as follows. The normal bundle of E is Ω^1_q, so $\omega_i = \omega_q$, so

$$\begin{aligned} R = q_* i^! &= q_*(\omega_i[-n] \otimes i^* -) \\ &\qquad\qquad = q_*(\omega_q[n][-2n] \otimes i^* -) = q_! i^*[-2n] = H^n L. \end{aligned}$$

Let $p : \Omega^1_q \to E$ be the projection; then for any $\mathcal{F} \in D^b(E)$ we have

$$\begin{aligned} i^! i_* \mathcal{F} &= i^! i_* i^* p^* \mathcal{F} \\ &= i^!(p^* \mathcal{F} \otimes i_* \mathcal{O}_E) \\ &= i^* p^* \mathcal{F} \otimes i^! i_* \mathcal{O}_E \\ &= \mathcal{F} \otimes i^! i_* \mathcal{O}_E. \end{aligned}$$

Moreover,

$$\begin{aligned} i^! i_* \mathcal{O}_E &= \omega_i[-n] \otimes i^* i_* \mathcal{O}_E \\ &= \omega_q[-n] \otimes (\mathcal{O}_E \oplus T_q[1] \oplus \Lambda^2 T_q[2] \oplus \cdots \oplus \Lambda^n T_q[n]) \\ &= \mathcal{O}_E \oplus \Omega^1_q[-1] \oplus \Omega^2_q[-2] \oplus \cdots \oplus \Omega^n_q[-n]. \end{aligned}$$

Thus for any $\mathcal{G} \in D^b(E)$ we have

$$\begin{aligned} RF\mathcal{F} &= q_* i^! i_* q^* \mathcal{G} \\ &= q_*(q^* \mathcal{G} \otimes i^! i_* \mathcal{O}_E) \\ &= \mathcal{G} \otimes q_*(\mathcal{O}_E \oplus \Omega^1_q[-1] \oplus \Omega^2_q[-2] \oplus \cdots \oplus \Omega^n_q[-n]) \\ &= \mathcal{G} \otimes (\mathcal{O}_Z \oplus \mathcal{O}_Z[-2] \oplus \mathcal{O}_Z[-4] \oplus \cdots \oplus \mathcal{O}_Z[-2n]) \end{aligned}$$

since $q_* \Omega^k_q = \mathcal{O}_Z[-k]$.

If B is a point then the isomorphism $\operatorname{Ext}^*(\mathcal{O}_P, \mathcal{O}_P) \cong H^*(\mathbb{P}^n)$ is well-known to be a ring isomorphism. For more general B, note that the zeros below the diagonal in hypothesis (b) come for free, since $\operatorname{Ext}^{<0}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[i]) = 0$. Thus we need only check that $RF[-2] \to RFRF \xrightarrow{R\epsilon F} RF$ induces an isomorphisms on $\mathcal{H}^i$ for $2 \le i \le 2n-2$. This can be done pointwise, where it follows from the well-known case.

5. The following example is due to Kawamata [24]. Let X be a 3-fold with an A_2 singularity, $\tilde{X} \to X$ the blowup of the singular point, E the exceptional divisor, l a ruling of the quadric cone E, and $\mathcal{B} =$

${}^\perp\langle \mathcal{O}_E(E)\rangle \subset D^b(\tilde{X})$. Then he shows that $\mathrm{Perf}(X)^\perp \subset \mathcal{B}$ is generated by one object $\mathcal{E} = \mathcal{O}_E(-l)$, that $\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \cong \mathbb{C}[h]/h^3$ as graded rings with $\deg h = 1$, and that the Serre functor $S_\mathcal{B}$ acts as $S_\mathcal{B}\mathcal{E} \cong \mathcal{E}[2]$. Thus the functor $\mathcal{E} \otimes - : D^b(\text{point}) \to \mathcal{B}$ is a $\mathbb{P}^2$-functor with $H = [-1]$.

6. Examples like the previous one, which we might sheepishly call $\mathbb{RP}^n$-objects, are equivalent to Toda's fat spherical objects with $A = \mathbb{C}[\epsilon]/\epsilon^2$ the ring of dual numbers, as follows. Let $\mathcal{E} \in D^b(X)$ be an object such that $\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) = \mathbb{C}[h]/h^{n+1}$ with $\deg h = 1$ instead of 2. Let $\mathcal{E}' \in D^b(\mathrm{Spec}\, A \times X)$ be the first-order deformation corresponding to $h \in \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ and $\pi : \mathrm{Spec}\, A \times X \to X$; then $\pi_*\mathcal{E}'$ is the non-trivial extension
$$0 \to \mathcal{E} \to \pi_*\mathcal{E}' \to \mathcal{E} \to 0.$$
Apply $\mathrm{Hom}(\mathcal{E}, -)$ to get
$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}(\pi_*\mathcal{E}',\mathcal{E}) \to \mathrm{Hom}(\mathcal{E},\mathcal{E}) \\ &\xrightarrow{\cdot h} \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^1(\pi_*\mathcal{E}',\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \\ &\xrightarrow{\cdot h} \cdots \\ &\xrightarrow{\cdot h} \mathrm{Ext}^n(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^n(\pi_*\mathcal{E}',\mathcal{E}) \to \mathrm{Ext}^n(\mathcal{E},\mathcal{E}) \to 0. \end{aligned}$$
Then the boundary maps are all isomorphisms, so $\mathrm{Ext}^*(\pi_*\mathcal{E}',\mathcal{E}) \cong H^*(S^n,\mathbb{C})$, so $\mathcal{E}'$ is a fat spherical object. From the same long exact sequence we see that the converse holds: if $\mathrm{Ext}^*(\pi_*\mathcal{E}',\mathcal{E}) \cong H^*(S^n,\mathbb{C})$ then $\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \cong \mathbb{C}[h]/h^{n+1}$ as algebras. We will see that the $\mathbb{P}^n$-twist defined below coincides with the fat spherical twist associated to $\mathcal{E}^*$.

Thus, for example, let X be a 3-fold and $C \subset X$ a $(0,-2)$-curve which deforms to first order but not to second order; then the functor $\mathcal{O}_C \otimes - : D^b(\text{point}) \to D^b(X)$ is a $\mathbb{P}^3$-functor with $H = [-1]$.

3.3 Construction of the $\mathbb{P}$-Twist

We first recall Huybrechts and Thomas's definition the $\mathbb{P}$-twist associated to a $\mathbb{P}$-object $\mathcal{E}$. Let $h : \mathcal{E}[-2] \to \mathcal{E}$ be map corresponding to a generator of $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$, and $h^* : \mathcal{E}^*[-2] \to \mathcal{E}^*$ the transpose. Then the $\mathbb{P}$-twist is the functor $P : D^b(X) \to D^b(X)$ induced by the double cone
$$\mathrm{cone}(\mathrm{cone}(\mathcal{E}^* \boxtimes \mathcal{E}[-2] \xrightarrow{h^*\boxtimes \mathrm{id} - \mathrm{id}\boxtimes h} \mathcal{E}^* \boxtimes \mathcal{E}) \xrightarrow{\mathrm{eval}} \mathcal{O}_\Delta)$$

in $D^b(X \times X)$. Since $\operatorname{Ext}^{-1}(\mathcal{E}^* \boxtimes \mathcal{E}[-2], \mathcal{O}_\Delta) = \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) = 0$, there is a unique way to take this double cone. For us there will be a canonical but non-unique way to do it.

To define the $\mathbb{P}$-twist associated to a $\mathbb{P}$-functor F, first let $j : H \to RF$ be the map coming from the splitting (3.1) let f be the composition

$$FHR \xrightarrow{FjR} FRFR \xrightarrow{\epsilon FR - FR\epsilon} FR$$

(recall that $\epsilon : FR \to \mathrm{id}_{\mathcal{B}}$ is the counit of the adjunction). Then f replaces $h^* \boxtimes \mathrm{id} - \mathrm{id} \boxtimes h$. The composition

$$FHR \xrightarrow{f} FR \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{B}}$$

is zero, so we can take the double cone. Note that

$$\operatorname{Ext}^{-1}(FHR, \mathrm{id}_{\mathcal{B}}) = \operatorname{Ext}^{-1}(H, RF)$$

need not vanish; for example, if $H = [-2]$, this Ext group is $HH^1(\mathcal{A})$, and if $H = [-1]$ it is $HH^0(\mathcal{A})$, which never vanishes.

The functors $T = \operatorname{cone} \epsilon$ and $C = \operatorname{cone} \eta$, which were equivalences when F was spherical, will now be used in an auxiliary way. We wish to produce a lift $\tilde{f}$ as in the diagram

$$\begin{tikzcd}
 & T[-1] \arrow[d] \\
FHR \arrow[r, "f"'] \arrow[ur, dotted, "\tilde{f}"] & FR \arrow[d, "\epsilon"] \\
 & \mathrm{id}_{\mathcal{B}}
\end{tikzcd}$$

and define $P = \operatorname{cone} \tilde{f}[1]$; using the octahedral axiom one can check that this is the same as

$$\operatorname{cone}(\operatorname{cone}(FHR \xrightarrow{f} FR) \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{B}}).$$

To produce $\tilde{f}$, we will lift

$$\begin{tikzcd}
 & T[-1] \arrow[d] \\
FRFR \arrow[r, "\epsilon FR - FR\epsilon"'] \arrow[ur, dotted] & FR
\end{tikzcd} \tag{3.2}$$

using the splitting of RFR discussed in §1.3. Consider the diagram

$$\begin{tikzcd}
FRT[-1] \arrow[r, "\epsilon T[-1]"] \arrow[d] \arrow[dd, bend right=60, "\cong"'] & T[-1] \arrow[d] \\
FRFR \arrow[r, "\epsilon FR - FR\epsilon"'] \arrow[d] & FR \\
FCR. &
\end{tikzcd}$$

The identity $FRFR \to FRFR$ is the sum of the two idempotents

$$FRFR \to FCR \xleftarrow{\cong} FRT[-1] \to FRFR$$
$$FRFR \xrightarrow{FR\epsilon} FR \xrightarrow{F\eta R} FRFR,$$

and the composition

$$FR \xrightarrow{F\eta R} FRFR \xrightarrow{\epsilon FR - FR\epsilon} FRFR$$

is zero, so $FRFR \xrightarrow{\epsilon FR - FR\epsilon} FRFR$ equals the composition

$$FRFR \to FCR \xleftarrow{\cong} FRT[-1] \to FRFR \xrightarrow{\epsilon FR - FR\epsilon} FRFR.$$

Thus the composition

$$FRFR \to FCR \xleftarrow{\cong} FRT[-1] \xrightarrow{\epsilon T[-1]} T[-1]$$

gives lift in (3.2).

Definition. If F is a $\mathbb{P}$-functor, the associated $\mathbb{P}$-*twist* is the cone P on the following composition:

$$P := \mathrm{cone}(FHR[1] \xrightarrow{FjR[1]} FRFR[1] \to FCR[1] \xleftarrow{\cong} FRT \xrightarrow{\epsilon T} T).$$

In Example 3 above we claimed that for $\mathbb{P}^1$-functors, which are the same as split spherical functors, the $\mathbb{P}^1$-twist is the square of the spherical twist. To see this, observe that the composition $FHR \to FRFR \to FCR$ is an isomorphism in this case, so

$$P = \mathrm{cone}(FRT \xrightarrow{\epsilon T} T) = TT$$

since $T = \mathrm{cone}\,\epsilon$.

In Example 6 we claimed that for an "$\mathbb{RP}^n$-object" $\mathcal{E}$, the fat spherical twist associated to $\mathcal{E}$ coincides with the $\mathbb{P}^n$-twist associated to $\mathcal{E}^*$. Let $\mathcal{E}$ and $\mathcal{E}'$ be as in that example and let $h : \mathcal{E} \to \mathcal{E}[1]$ correspond to $h \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})$, so we have an exact triangle

$$\mathcal{E} \to \pi_* \mathcal{E}' \to \mathcal{E} \xrightarrow{h} \mathcal{E}[1].$$

Let $h^* : \mathcal{E}[-1] \to \mathcal{E}$ be its transpose, so we have an exact triangle

$$\mathcal{E}^* \to \pi_*(\mathcal{E}'^*) \to \mathcal{E}^* \xrightarrow{-h^*} \mathcal{E}^*[1],$$

where the minus sign is due to the shift. If $F : D^b(\operatorname{Spec} A) \to D^b(X)$ is the functor induced by $\mathcal{E}'$ and R its right adjoint then FR is induced by

$$(\pi_{13*}(\pi_{12}^* \mathcal{E}'^* \otimes \pi_{23}^* \mathcal{E}'))^* \in D^b(X \times X),$$

where π_{ij} are the projections from $X \times \operatorname{Spec} A \times X$. Now $\pi_{23}^* \mathcal{E}' \in D^b(X \times \operatorname{Spec} A \times X)$ is a first-order deformation of $\mathcal{O}_X \boxtimes \mathcal{E}$, and $\pi_{12}^* \mathcal{E}'^*$ is a deformation of $\mathcal{E}^* \boxtimes \mathcal{O}_X$, so $\pi_{12}^* \mathcal{E}'^* \otimes \pi_{23}^* \mathcal{E}'$ is a deformation of $\mathcal{E}^* \boxtimes \mathcal{E}$, and when we push down we get an exact triangle

$$\mathcal{E}^* \boxtimes \mathcal{E} \to \underbrace{\pi_{13*}(\pi_{12}^* \mathcal{E}'^* \otimes \pi_{23}^* \mathcal{E}')}_{=(FR)^*} \to \mathcal{E}^* \boxtimes \mathcal{E} \xrightarrow{\mathrm{id} \boxtimes h - h^* \boxtimes \mathrm{id}} \mathcal{E}^* \boxtimes \mathcal{E}[1].$$

Thus we have

$$\begin{aligned} T &= \operatorname{cone}(FR \to \mathrm{id}) \\ &= \operatorname{cone}(\operatorname{cone}(\mathcal{E} \boxtimes \mathcal{E}^*[-1] \xrightarrow{\mathrm{id} \boxtimes h^* - h \boxtimes \mathrm{id}} \mathcal{E} \boxtimes \mathcal{E}^*) \to \mathrm{id}) \end{aligned}$$

as claimed.

3.4 Action on Cohomology

If $\mathcal{E}$ is a $\mathbb{P}^n$-object then Huybrechts and Thomas show that the $\mathbb{P}^n$-twist P sends $\mathcal{E}$ to $\mathcal{E}[-2n]$ and acts as the identity on $\mathcal{E}^\perp$. The action on cohomology is trivial: the cone on

$$\mathcal{E}^* \boxtimes \mathcal{E}[-2] \to \mathcal{E}^* \boxtimes \mathcal{E}$$

is zero in K-theory, so $\mathcal{O}_\Delta$ and the kernel inducing P are the same in K-theory.

For a $\mathbb{P}^n$-functor $F : \mathcal{A} \to \mathcal{B}$, this is generalized as follows. Again the, the $\mathbb{P}$-twist

$$P = \operatorname{cone}(FHR[1] \to T)$$

acts as the identity on $(\operatorname{im} F)^\perp = \ker R$: if $B \in \ker R$ then $PB = TB = B$. To see how it acts on $\operatorname{im} F$, we need the following:

Proposition. $PF \cong FH^{n+1}[-2]$

Proof. We have

$$PF[-1] = \operatorname{cone}(FHRF \xrightarrow{FjRF} FRFRF \dashrightarrow FRTF[-1] \xrightarrow{\epsilon TF[-1]} TF[-1]),$$

where the dashed arrow comes from (3.2). We can simplify as follows: the diagram

$$\begin{tikzcd}
FRTF[-1] \arrow[r, "\epsilon TF[-1]"] \arrow[d] & TF[-1] \arrow[d] \\
FRFRF \arrow[r, "\epsilon FR"'] \arrow[d] & FR \arrow[d] \\
FRFC \arrow[r, "\epsilon FC"'] & FC
\end{tikzcd}$$

commutes, and the vertical compositions are isomorphisms, so

$$PF[-1] = \operatorname{cone}(FHRF \xrightarrow{FjRF} FRFRF \xrightarrow{\epsilon FRF} FRF \to FC).$$

But

$$FHRF = F(H \oplus H^2 \oplus \cdots \oplus H^n \oplus H^{n+1}),$$
$$FC = F(H \oplus H^2 \oplus \cdots \oplus H^n),$$

and our hypothesis (b) implies that the cone on this is $FH^{n+1}[1]$, as desired. ∎

Thus for example $H = [-2]$ then P acts on $\operatorname{im} F$ by $[-2n]$, or if $H = [-1]$ then P acts on $\operatorname{im} F$ by $[-n+1]$.

If $F : D^b(X) \to D^b(Y)$ is a $\mathbb{P}^n$-functor with $H = [-2]$ then P acts trivially on cohomology, but if $H = [-1]$ or something more interesting then the action may be more interesting.

Now we can emulate the end of §1.4 to show that if S is a K3-surface and $F : D^b(S) \to D^b(S^{[n]})$ a $\mathbb{P}^{[n-1]}$-functor with $H = [-2]$, the $\mathbb{P}$-twist P_F associated to F is different from the known autoequivalences of $D^b(S^{[n]})$. We know that P_F shifts $\operatorname{im} F$ by $-2n+2$ and fixes $\ker R$. If $\mathcal{E} \in D^b(S)$ is a spherical object with $\mathcal{E}^\perp \neq \emptyset$, $T_\mathcal{E}$ the spherical twist associated to $\mathcal{E}$, and $\varphi : \operatorname{Aut}(D(S)) \hookrightarrow \operatorname{Aut}(D(S^{[n]})$ Ploog's map, then there non-zero objects which $\varphi(T_\mathcal{E})$ shifts by $-2n, -2n+1, \ldots, -2, -1$, and 0, so P_F is not a shift

of $\varphi(T_{\mathcal{E}})$. If $\mathcal{E} \in D^b(S^{[n]})$ is a $\mathbb{P}^n$-object with $\mathcal{E}^\perp \neq 0$[9] then $P_{\mathcal{E}}$ shifts $\mathcal{E}$ by $-2n$ and fixes $\mathcal{E}^\perp$, so P_F is not a shift of $P_{\mathcal{E}}$.

3.5 Proof of Equivalence

Theorem 3. *If $\mathcal{B}$ is indecomposable then P is an equivalence.*

Proof. We model our proof on that of Theorem 1. Again we consider the spanning class $\Omega = \ker R \cup \operatorname{im} F$.

First, if $B, B' \in \ker R$ then $\operatorname{Hom}(PB, PB') = \operatorname{Hom}(B, B')$. Next, if $FA \in \operatorname{im} F$ and $B \in \ker R = \ker L$ then

$$\begin{aligned}\operatorname{Hom}(PFA, PB) &= \operatorname{Hom}(FH^{n+1}A[2], B)\\ &= \operatorname{Hom}(H^{n+1}A[2], RB) = 0 = \operatorname{Hom}(FA, B)\end{aligned}$$

$$\begin{aligned}\operatorname{Hom}(PB, PFA) &= \operatorname{Hom}(B, FH^{n+1}A[2])\\ &= \operatorname{Hom}(LB, H^{n+1}A[2]) = 0 = \operatorname{Hom}(B, FA).\end{aligned}$$

Last, if $FA, FA' \in \operatorname{im} F$ then

$$\begin{aligned}\operatorname{Hom}(PFA, PFA') &= \operatorname{Hom}(FH^{n+1}A, FH^{n+1}B)\\ &= \operatorname{Hom}(A, H^{-n-1}RFH^{n+1}B) = \operatorname{Hom}(A, RFB) = \operatorname{Hom}(FA, FB).\end{aligned}$$

Thus P is fully faithful. To show that P is an equivalence, we show that $\ker P^l = 0$. Take left adjoints of $PF \cong H^{n+1}F[2]$ to get $H^{-n-1}L[-2] \cong LP^l$, so if $P^l B = 0$ then $H^{-n-1}LB = 0$, so $LB = 0$. Take left adjoints of the definition of P to get

$$P^l[1] = \operatorname{cone}(T^l \to FH^{-1}L[-1]),$$

from which we see that if $P^l B = 0$ then $T^l B = 0$, so $B = 0$ as in the proof of Theorem 1. $\square$

[9] Again, this holds for all known $\mathcal{E}$: if $\mathcal{E}$ is a line bundle, consider $\mathcal{E} \otimes F\mathcal{O}_x^* \otimes F\mathcal{O}_y$, where $x, y \in S$ are distinct points, and if $\mathcal{E}$ is the structure sheaf of a $\mathbb{P}^n \subset S^{[n]}$, consider the structure sheaf of a point not in the $\mathbb{P}^n$.

4 Cubic Fourfold Calculation

Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface, $Y \subset \mathrm{Gr}(2,6)$ the variety of lines on X,

$$L = \{(x,l) \in X \times Y : x \in l\}$$

the universal line, and $\mathcal{A} = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^\perp \subset D^b(X)$, that is,

$$\mathcal{A} = \{A \in D^b(X) : \mathrm{RHom}(\mathcal{O}_X(t), A) = 0 \text{ for } t = 0,1,2\}.$$

Let $i : \mathcal{A} \to D^b(X)$ be the inclusion; since the three line bundles form an exceptional collection, i has left and right adjoints i^l and i^r, that is, $\mathcal{A}$ is an admissible subcategory. Let $X \xleftarrow{p} L \xrightarrow{q} Y$ be the projections. We will consider the functor $F := q_* p^* i : \mathcal{A} \to D^b(Y)$ and its right adjoint $R = i^r q^! p_*$.

In §4.1 we explain that Y is a moduli space of sheaves in $\mathcal{A}$ and that F is essentially the functor induced by the universal sheaf. In §4.2 we prove:

Theorem 4. *The composition $RF = \mathrm{id}_\mathcal{A} \oplus [-2]$, so F is spherical.*

In §4.3 show that the associated spherical twist sends the structure sheaf of a point to a complex of rank 2, so it is not generated by the previously-known autoequivalences of $D^b(Y)$. Presumably this complex is a reflexive sheaf, locally free away from the original point, but we do not show this.

4.1 Y as a Moduli Space of Sheaves in $\mathcal{A}$

Let l be a line on X and y the corresponding point of Y, and let us describe $R\mathcal{O}_y$.

Since $q : L \to Y$ is a $\mathbb{P}^1$-bundle and p maps the fibers of q to straight lines in X, we have $\omega_q = p^*\mathcal{O}_X(-2) \otimes q^*\mathcal{L}$ for some line bundle $\mathcal{L}$ on Y—in fact $\mathcal{L}$ is the restriction to Y of the Plücker line bundle on $\mathrm{Gr}(2,6)$, but we will not need this. Thus

$$R\mathcal{O}_y = i^r p_* q^! \mathcal{O}_y = i^r \mathcal{O}_l(-2)[1] = S_\mathcal{A} i^l S_X^{-1} \mathcal{O}_l(-2)[1] = i^l \mathcal{O}_l(1)[-1],$$

where we have used the fact $\omega_X = \mathcal{O}_X(-3)$. Now i^l is just given by mutating past $\mathcal{O}_X(2)$, $\mathcal{O}_X(1)$, and $\mathcal{O}_X$. We already have $\mathrm{Ext}^*(\mathcal{O}_X(2), \mathcal{O}_l(1)) = 0$, so the first mutation does nothing. The second mutation sends $\mathcal{O}_l(1)[-1]$ to the twisted ideal sheaf $\mathcal{I}_l(1)$.

For the third mutation, let $\mathcal{F}_l$ be the "second syzygy sheaf" defined by the exact sequence

$$0 \to \mathcal{F}_l \to \mathcal{O}_X(-1)^4 \to \mathcal{O}_X \to \mathcal{O}_l \to 0,$$

where for example if l is the line $x_0 = x_1 = x_2 = x_3 = 0$ then the map $\mathcal{O}_X(-1)^4 \to \mathcal{O}_X$ is given by the matrix $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}$. Then $\mathcal{F}_l$ is a reflexive sheaf of rank 3, locally free away from l, and mutation past $\mathcal{O}_X$ sends $\mathcal{I}_l(1)$ to $\mathcal{F}_l(1)[1]$.

Kuznetsov and Markushevich [30, §5] show that $\mathcal{F}_l$ is stable, that $\mathcal{F}_l \ncong \mathcal{F}_{l'}$ when $l \neq l'$, and that the natural map $T_y Y \to \operatorname{Ext}^1(\mathcal{F}_l, \mathcal{F}_l)$ is an isomorphism, so the map $y \mapsto \mathcal{F}_l$ gives an isomorphism of Y with a component of the moduli space of sheaves on X with the same Chern classes as $\mathcal{F}_l$. In fact, since $S_{\mathcal{A}} = [2]$, the proof of [23, Thm. 4.1] is easily adapted to show that this moduli space is connected, so Y is the whole moduli space.

Now if Y were a moduli space of sheaves on a K3 surface S and and $F : D^b(S) \to D^b(Y)$ the functor induced by the universal sheaf $\mathcal{U}$ on $S \times Y$ then the right adjoint R would be induced by $\mathcal{U}^*[2]$. In this case, the derived duals of the sheaves $\mathcal{F}_l$ on X are complexes, not sheaves, but this does not bother us.

4.2 F is Spherical

Before proving that $RF = \mathrm{id}_{\mathcal{A}} \oplus [-2]$, let us say how this implies that F is spherical. Roughly, the unit map $\eta : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}} \oplus [-2]$ is just $1 \oplus 0$ because

$$\begin{aligned}
\operatorname{Hom}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}}) &= \operatorname{Ext}^{-2}(S_{\mathcal{A}}^{-1}, \mathrm{id}_{\mathcal{A}}) = HH_2(\mathcal{A}) = HH_2(X) = H^{3,1}(X) = \mathbb{C} \\
\operatorname{Hom}(\mathrm{id}_{\mathcal{A}}, [-2]) &= \operatorname{Ext}^{-4}(S_{\mathcal{A}}^{-1}, \mathrm{id}_{\mathcal{A}}) = HH_4(\mathcal{A}) = HH_4(X) = 0,
\end{aligned}$$

where we have used the fact that $S_{\mathcal{A}} = [2]$, that

$$\begin{aligned}
HH_i(X) &= HH_i(\mathcal{A}) \oplus HH_i(\langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle) \\
&= HH_i(\mathcal{A}) \oplus HH_i(3 \text{ points}),
\end{aligned}$$

and that the Hodge diamond of X is

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1. & & & &
\end{array}$$

This rough argument is actually valid because we are working with Fourier–Mukai kernels, not just functors; everything is proved rigorously in [26]. Thus the the cotwist $C = \operatorname{cone} \eta = [-2]$, which is an isomorphism. Since $S_{\mathcal{A}} = [2]$ and $S_Y = [4]$, the requirement $S_Y F C \cong F S_{\mathcal{A}}$ is satisfied.

Now our main task is to describe the composition $p_* q^! q_* p^*$. Consider the diagram

$$\begin{tikzcd}
 & & L \times_Y L \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
 & L \arrow[dl, "p"] \arrow[dr, "q"'] & & L \arrow[dl, "q"] \arrow[dr, "p"'] & \\
X & & Y & & X
\end{tikzcd}$$

Since q is a $\mathbb{P}^1$ bundle, it is flat, so

$$\begin{aligned}
p_* q^! q_* p^* - &= p_*(\omega_q \otimes q^* q_* p^* -)[1] \\
&= p_*(\omega_q \otimes \pi_{2*} \pi_1^* p^* -)[1] \\
&= (p\pi_2)_*(\pi_2^* \omega_q \otimes (p\pi_1)^* -)[1]
\end{aligned}$$

Thus if we let $\psi = p\pi_1 \times p\pi_2 : L \times_Y L \to X \times X$, then the functor $p_* q^! q_* p^*$ is induced by

$$\psi_* \pi_2^* \omega_q[1] \in D^b(X \times X). \tag{4.1}$$

Consider the Beilinson resolution of the diagonal $\Delta_L \subset L \times_Y L$:

$$0 \to \mathcal{O}_q(-1) \boxtimes_Y \omega_q(1) \to \mathcal{O}_{L \times_Y L} \to \mathcal{O}_{\Delta_L} \to 0.$$

We have argued that $p^* \mathcal{O}_X(1) = \mathcal{O}_q(1)$, so

$$\mathcal{O}(-\Delta_L) = \psi^* \mathcal{O}_{X \times X}(-1, 1) \otimes \pi_2^* \omega_q.$$

Thus the object (4.1) that we want to compute can be rewritten as

$$\mathcal{O}_{X \times X}(1, -1) \otimes \psi_* \mathcal{O}(-\Delta_L)[1].$$

Let $Z = \psi(L \times_Y L) \subset X \times X$. It is the closure of the set

$$\{(x, y) \in X \times X : x \neq y \text{ and the line } \overline{xy} \text{ lies in } X\}.$$

To get equations for Z, let f be a polynomial defining X and $\hat{f}$ the polarization of f, that is, the unique symmetric trilinear form with $\hat{f}(v, v, v) = f(v)$ for all $v \in \mathbb{C}^6$. Given distinct points $x, y \in X$, the line $\overline{xy}$ lies in X if and only if

$$\begin{aligned}
0 &= f(ax + by) \\
&= a^3 \hat{f}(x, x, x) + 3a^2 b \hat{f}(x, x, y) + 3ab^2 \hat{f}(x, y, y) + b^3 \hat{f}(y, y, y)
\end{aligned}$$

for all $a, b \in \mathbb{C}$, which is true if and only if

$$\hat{f}(x, x, y) = 0$$
$$\hat{f}(x, y, y) = 0.$$

(If you like, the first equation says that the line is tangent to X at x, and the second that it is tangent at y.) Thus Z is the complete intersection of hypersurfaces of bidegrees $(2, 1)$ and $(1, 2)$ in $X \times X$.

First I claim that $R^0\psi_*\mathcal{O}_{L\times_Y L} = \mathcal{I}_{\Delta_X/Z}$. Observe that ψ is an isomorphism away from Δ_L, hence is birational. Now Z is regular in codimension 1, since $Z \setminus \Delta_X \cong (L \times_Y L) \setminus \Delta_L$, and it satisfies Serre's condition S2, being a complete intersection, so it is normal. Thus by Zariski's Main Theorem, ψ has connected fibers, so $R^0\psi_*\mathcal{O}_{L\times_Y L} = \mathcal{O}_Z$ and $R^0\psi_*\mathcal{O}_{\Delta_L} = \mathcal{O}_{\Delta_X}$, which proves the claim.

Next we study $R^j\psi_*\mathcal{O}(-\Delta_L)$ for $j > 0$ using the Grothendieck spectral sequence

$$E_2^{ij} = R^i\varpi_{2*}R^j\psi_*\mathcal{O}(-\Delta_L) \Rightarrow R^{i+j}(\varpi_2\psi)_*\mathcal{O}(-\Delta_L) \tag{4.2}$$

where $\varpi_2 : X \times X \to X$ is projection onto the second factor. The right-hand side vanishes, as follows: the diagram

$$\begin{tikzcd} L \times_Y L \arrow[r, "\pi_2"] \arrow[d, "\psi"'] & L \arrow[d, "p"] \\ X \times X \arrow[r, "\varpi_2"'] & X \end{tikzcd}$$

commutes, and $\pi_{2*}\mathcal{O}(-\Delta_L) = 0$ because π_2 is a $\mathbb{P}^1$-bundle. Next, we see that $R^j\psi_*\mathcal{O}(-\Delta_L)$ is supported on Δ_X for $j > 0$, at least set-theoretically, because ψ is finite away from Δ_L. Next, from the Koszul resolution

$$0 \to \mathcal{O}(-3, -3) \to \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) \to \mathcal{O}_{X\times X} \to \mathcal{O}_Z \tag{4.3}$$

and the fact that $\omega_X = \mathcal{O}_X(-3)$ we find that

$$R^i\varpi_{2*}\mathcal{O}_Z = \begin{cases} \mathcal{O}_X & i = 0 \\ \omega_X & i = 2 \\ 0 & \text{otherwise,} \end{cases}$$

so from the exact sequence

$$0 \to \mathcal{I}_{\Delta_X/Z} \to \mathcal{O}_Z \to \mathcal{O}_{\Delta_X} \to 0$$

we find that
$$R^i\varpi_{2*}\mathcal{I}_{\Delta_X/Z} = \begin{cases}\omega_X & i=2\\ 0 & \text{otherwise.}\end{cases}$$
Thus (4.2) becomes

$$\begin{tikzcd}
R^0\varpi_{2*}R^2\psi_*\mathcal{O}(-\Delta_L) \arrow[drr] & 0 & 0 \\
R^0\varpi_{2*}R^1\psi_*\mathcal{O}(-\Delta_L) \arrow[drr] & 0 & 0 \\
0 & 0 & \omega_X,
\end{tikzcd}$$

so $R^1\psi_*\mathcal{O}(-\Delta_L) = \omega_{\Delta_X}$ and $R^{\geq 2}\psi_*\mathcal{O}(-\Delta_L) = 0$.

Thus we have an exact triangle
$$\mathcal{I}_{\Delta_X/Z} \to \psi_*\mathcal{O}(-\Delta_X) \to \omega_{\Delta_X}[-1].$$
Tensoring with $\mathcal{O}(1,-1)[1]$ we get an exact triangle
$$\mathcal{I}_{\Delta_X/Z}(1,-1)[1] \to p_*q^!q_*p^* \to S_X[-4] \tag{4.4}$$
which we want to compose with i on the right and i^r on the left.

Take the exact triangle
$$\mathcal{I}_{\Delta_X/Z} \to \mathcal{O}_Z \to \mathcal{O}_{\Delta_X},$$
rotate and tensor with $\mathcal{O}(1,-1)[1]$ to get
$$\mathcal{O}_{\Delta_X} \to \mathcal{I}_{\Delta_X/Z}(1,-1)[1] \to \mathcal{O}_Z(1,-1)[1].$$
From (4.3) we see that $\mathcal{O}_Z(1,-1)$ is quasi-isomorphic to the complex
$$\mathcal{O}(-2,-4) \to \mathcal{O}(-1,-2)\oplus\mathcal{O}(0,-3) \to \mathcal{O}(1,-1).$$
Thus applying the functor induced by $\mathcal{O}_Z(1,-1)$ to an object iA, where $A \in \mathcal{A}$, we get a complex
$$\begin{aligned}
&\mathrm{R}\Gamma(iA(-2))\otimes\mathcal{O}_X(-4)\\
&\qquad\to \mathrm{R}\Gamma(iA(-1))\otimes\mathcal{O}_X(-2)\oplus\mathrm{R}\Gamma(iA)\otimes\mathcal{O}_X(-3)\\
&\qquad\qquad\to \mathrm{R}\Gamma(iA(1))\otimes\mathcal{O}_X(-1).
\end{aligned}$$

But $\mathrm{R}\Gamma(iA(-2)) = \mathrm{RHom}(\mathcal{O}_X(2), iA) = 0$, and similarly $\mathrm{R}\Gamma(iA(-1)) = \mathrm{R}\Gamma(iA) = 0$, so our complex is just

$$\mathrm{R}\Gamma(iA(1)) \otimes \mathcal{O}_X(-1).$$

Moreover,

$$i^r \mathcal{O}_X(-1) = S_{\mathcal{A}} i^l S_X^{-1} \mathcal{O}_X(-1) = i^l \mathcal{O}_X(2)[-2] = 0,$$

so $i^r \circ \mathcal{O}_Z(1,-1) \circ i = 0$, so

$$i^r \circ \mathcal{I}_{\Delta_X/Z}(1,-1)[1] \circ i = i^r \circ \mathcal{O}_{\Delta_X} \circ i = i^r i = \mathrm{id}_{\mathcal{A}}.$$

Lastly,

$$i^r S_X i[-4] = S_{\mathcal{A}}[-4] = [-2],$$

so composing (4.4) with i on the right and i^r on the left we get an exact triangle

$$\mathrm{id}_{\mathcal{A}} \to i^r p_* q^! q_* p^* i \to [-2],$$

which is split because

$$\mathrm{Ext}^1([-2], \mathrm{id}_{\mathcal{A}}) = \mathrm{Ext}^1(S_{\mathcal{A}}^{-1}, \mathrm{id}_{\mathcal{A}}) = HH_{-1}(\mathcal{A}) = HH_{-1}(X) = 0.$$

We mention again that this Hochschild argument is rigorous thanks to [26].

4.3 Twist of $\mathcal{O}_y$

In this subsection we show that if $y \in Y$ then $T\mathcal{O}_y$ is a complex of rank 2, where $T = \mathrm{cone}(FR \to \mathrm{id}_Y)$ is the spherical twist associated to our spherical functor F. In §4.1 we saw that $iR\mathcal{O}_y$ was the complex

$$\mathcal{O}_X^4 \to \underline{\mathcal{O}_X(1)} \to \mathcal{O}_l(1),$$

where the underlined term is in degree zero and $l \subset X$ is the line corresponding to y. Thus

$$\begin{aligned} \mathrm{rank}(T\mathcal{O}_y) &= \mathrm{rank}(\mathcal{O}_y) - \mathrm{rank}(FR\mathcal{O}_y) \\ &= 4\,\mathrm{rank}(q_* p^* \mathcal{O}_X) - \mathrm{rank}(q_* p^* \mathcal{O}_X(1)) + \mathrm{rank}(q_* p^* \mathcal{O}_l(1)). \end{aligned}$$

Since q is a $\mathbb{P}^1$-bundle we have $q_* p^* \mathcal{O}_X = \mathcal{O}_Y$, and since $p^* \mathcal{O}_X(1) = \mathcal{O}_q(1)$ we see that $q_* p^* \mathcal{O}_X(1)$ is a rank-2 vector bundle on Y, so it enough to show that $q_* p^* \mathcal{O}_l(1)$ is a complex of rank 0. We will argue that its support $q(p^{-1}(1))$ is a proper subvariety of Y because for any $x \in X$, the fiber

$p^{-1}(x) = \{l' \in Y : x \in l'\}$ is at most 2-dimensional. (The general fiber of p is a curve, and if X is general then p is flat, but for example if X is the Fermat cubic then p has some 2-dimensional fibers.)

A line $l' \subset X$ passing through x lies in the projective tangent space $\mathbb{T}_x X$ and is determined by where it meets a hyperplane H not containing x, so $p^{-1}(x)$ is isomorphic to a subvariety of $X \cap \mathbb{T}_x X \cap H$. Since X is smooth, it does not contain the hyperplane $\mathbb{T}_x X$, so $X \cap \mathbb{T}_x X$ is 3-dimensional. The singularities of $X \cap \mathbb{T}_x X$ are $\{y \in Y : \mathbb{T}_y X = \mathbb{T}_x X\}$. This is the fiber of the Gauss map, which is finite [19, p. 188], so $X \cap \mathbb{T}_x X$ is a 3-fold with isolated singular points, hence is irreducible. We see that $X \cap \mathbb{T}_x X \not\subset H$, so $X \cap \mathbb{T}_x X \cap H$ is a surface, which proves the claim.

A Appendix: Cohomology and Base Change

The following is well-known to those who know it well, but I could not find a reference.

Proposition. *Let X, Y, and B be connected schemes over a field k with X and Y Cohen–Macaulay and B smooth. Let $f : X \to B$ be proper and $g : Y \to B$ arbitrary, and $\tilde{f}$ and $\tilde{g}$ as in the diagram*

$$\begin{tikzcd} X \times_B Y \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

If every irreducible component of $X \times_B Y$ is of the expected dimension $\dim X + \dim Y - \dim B$ then the natural map $f^* g_* \to \tilde{g}_* \tilde{f}^*$ is an isomorphism.[10]

Proof. Let $\Gamma_f \subset X \times B$ be the graph of f and $\Gamma_g \subset B \times Y$ the graph of g. It is enough to show that on $X \times B \times Y$ we have $\operatorname{Tor}_i(\mathcal{O}_{\Gamma_f \times Y}, \mathcal{O}_{X \times \Gamma_g}) = 0$ for $i > 0$, a condition called "Tor-independence" [31, Thm. 3.10.3]. Note that $(\Gamma_f \times Y) \cap (X \times \Gamma_g) \cong X \times_B Y$.

First I claim that Γ_f is locally cut out of $X \times B$ by a regular sequence. Since B is smooth, the diagonal $\Delta \subset B \times B$ is locally cut out by a regular sequence of n functions, where $n = \dim B$. Thus $\Gamma_f = (f \times 1)^{-1}\Delta$ is locally cut out by n functions, which a priori may not be regular sequence; but

[10] For the reader who is just joining us we mention that all our functors are implicitly derived: we mean $Lf^* Rg_* \to R\tilde{g}_* L\tilde{f}^*$.

$X \times B$ is Cohen–Macaulay [43], so a sequence of n functions is regular if and only if it cuts out a subscheme of codimension n [35, Thm. 17.4(iii)], and the codimension of $\Gamma_f \cong X$ is indeed n.

Thus $\Gamma_f \times Y$ is locally cut out of $X \times B \times Y$ by a regular sequence of n functions, so locally we can resolve $\mathcal{O}_{\Gamma_f \times Y}$ by a Koszul complex. Tensoring with $\mathcal{O}_{X \times \Gamma_g}$, we see that the higher Tors vanish if the sequence remains regular when restricted to $X \times \Gamma_g$. Since $X \times \Gamma_g \cong X \times Y$ is Cohen–Macaulay and the subscheme $(\Gamma_f \times Y) \cap (X \times \Gamma_g) \cong X \times_B Y$ cut out by the restricted sequence has codimension n by hypothesis, we are done. $\square$

Remarks. 1. The dimension hypothesis is necessary: let B be a smooth surface, X a point, and Y the blowup of B at $f(X)$; then $X \times_B Y$ is the exceptional line $E \subset Y$, whose dimension is $1 > 0 + 2 - 2$, and one computes that $f^* g_* \mathcal{O}_E(E) = 0$ while $\tilde{g}_* \tilde{f}^* \mathcal{O}_E(E) = \mathcal{O}_X[1]$.

2. The smoothness of B is necessary: Let B be the cone $xy = z^2$ in $\mathbb{A}^3$, X the line $x = z = 0$, Y the line $y = z = 0$, and f and g the inclusions, so $X \times_B Y = X \cap Y$ is the origin, which is of the expected dimension. Using the resolution

$$\cdots \to \mathcal{O}_B^2 \xrightarrow{\left(\begin{smallmatrix} y & -z \\ -z & x \end{smallmatrix}\right)} \mathcal{O}_B^2 \xrightarrow{\left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right)} \mathcal{O}_B^2 \xrightarrow{\left(\begin{smallmatrix} y \\ -z \end{smallmatrix}\right)} \mathcal{O}_B \to g_* \mathcal{O}_Y \to 0$$

one computes that $\operatorname{Tor}_i(f_* \mathcal{O}_X, g_* \mathcal{O}_Y) = \mathcal{O}_{\text{origin}}$ for all $i \geq 0$. Thus $f_* f^* g_* \mathcal{O}_Y = f_* \mathcal{O}_X \otimes g_* \mathcal{O}_Y$ is different from $f_* \tilde{g}_* \tilde{f}^* \mathcal{O}_Y = f_* \tilde{g}_* \mathcal{O}_{X \cap Y}$.

3. The Cohen–Macaulay hypothesis is also necessary, as we see from the following example based on [13, Ex. 18.8]. Let $B = \mathbb{A}^4$. Let

$$X = \operatorname{Spec} k[s^4, s^3 t, s t^3, t^4]$$

be the affine cone over a rational curve of degree 4 in $\mathbb{P}^3$, which is not Cohen–Macaulay, and $f : X \to B$ the inclusion. Let Y be the plane $x = w = 0$ and g the inclusion. With Macaulay2 [16] one computes that that $X \cap Y$ is a scheme of length 5 supported at the origin, which is of the expected dimension, but $\operatorname{Tor}_1(f_* \mathcal{O}_X, g_* \mathcal{O}_Y) = \mathcal{O}_{\text{origin}}$, so again $f^* g_* \mathcal{O}_Y \neq \tilde{g}_* \tilde{f}^* \mathcal{O}_Y$.

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